p-adic Hodge Theory for Rigid Analytic Varieties

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Abstract

This thesis is an exposition of p-adic Hodge theory, with particular emphasis on the comparison theorems for rigid analytic varieties. The subject grew out of Tate's study of p-divisible groups [**Tat67**]. We will follow some of the ideas that led from his work to Scholze's comparison theorem of 2013 [**Sch13b**].

A central question in p-adic Hodge theory is how to describe the étale cohomology of a variety over a p-adic field in terms of its differentials. Tate obtained such a description for abelian varieties with good reduction [**Tat67**], and Faltings extended it to smooth proper algebraic varieties [**Fal88**]. What is particularly interesting about Scholze's proof is that it works for smooth proper rigid spaces, using the theory of perfectoid spaces. Our goal is to explain this proof.

We begin with the theory of p-adic representations, where we encounter the period rings, and then develop the analytic geometry needed to use these rings on a rigid space. Along the way, we will see why the pro-étale site is so useful in making these two theories work together. We also discuss some applications of the comparison theorems, including de Rham representations, Hodge numbers, and algebraic cycle classes, and conclude with a brief discussion of torsion in integral cohomology.

Acknowledgements

First and foremost, I would like to thank my advisor Sug Woo Shin for his guidance while writing this thesis, be it through answering questions, providing references, and support. I would also like to thank Andrew Cheng for comments on an earlier draft.

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CHAPTER 1

Introduction

Much of the motivation for p-adic Hodge theory comes from a beautiful theorem in classical Hodge theory. Roughly speaking, the theorem says that we can read off information about the topology of a complex manifold from its differential forms. The desire for a similar description of the cohomology of a variety over a p-adic field goes back to Tate. The aim of this thesis is to explain Scholze's 2013 result in the setting of smooth proper rigid analytic varieties.

1. Classical Hodge Theory

Classical Hodge theory has its roots in the work of several nineteenth-century mathematicians who were interested in evaluating integrals of the form

$$\int \frac{dx}{\sqrt{(x-z_1)\cdots(x-z_n)}}$$

over various contours in the complex plane. Such integrals are known as *period integrals*. Riemann's insight was to view these as integrals on a Riemann surface. This allowed one to approach the original question geometrically, by studying Riemann surfaces and eventually more general complex manifolds.

One of the remarkable things about the resulting theory is that it relates two rather different ways of studying a complex manifold: its topological cohomology and the cohomology of its sheaves of differentials. This is the *Hodge decomposition*.

THEOREM 1.1. *If X is a compact Kähler manifold, there is a canonical isomorphism*

$$H^n(X, \mathbf{Z}) \otimes \mathbf{C} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j) \tag{1.1}$$

In particular, for a Riemann surface, this gives a decomposition of the first cohomology group:

$$H^1(X, \mathbf{Z}) \otimes \mathbf{C} \cong H^0(X, \Omega^1) \oplus H^1(X, \Omega^0)$$

By Serre's GAGA theorems, we can interpret the right-hand side of this decomposition algebraically when the manifold comes from a smooth projective variety over $\mathbf{C}$. Once we phrase the result in this language, it is rather tempting to ask for a similar result over other fields. We will be interested in what happens over a p-adic field.

2. Hodge-Tate Decompositions

The first p-adic analogue of (1.1) appeared in Tate's work on p-divisible groups [Tat67]. If A is an abelian variety with good reduction over a p-adic field K, Tate found a relationship between the étale cohomology of $A_{\bar{K}}$ and the algebraic de Rham cohomology of A. There is a slight complication here that we did not have in the complex setting: the étale cohomology carries a natural Galois action, and we would like our comparison to respect it. This is where the so-called Tate twists come in.

DEFINITION 1.2. Let M be a $\mathbf{Z}_p$-module. The *n-th Tate twist* of M, denoted $M(n)$, is $M \otimes_{\mathbf{Z}_p} (T_p \mathbf{G}_m)^{\otimes n}$. If $n < 0$, we take duals instead. Here, $T_p \mathbf{G}_m$ denotes the p-adic Tate module of $\mathbf{G}_m$.

With the notion of a Tate twist, we can formulate the decomposition Tate obtained.

THEOREM 1.3. *Let K be a p-adic field, and let A be an abelian variety over K with good reduction. Let $\mathbf{C}_K$ be a completed algebraic closure of K, and let $A_{\bar{K}}$ denote the base change $A \times_K \bar{K}$. Then there is an isomorphism*

$$H^1_{\acute{e}t}(A_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_K \cong H^1(A, \mathcal{O}_A) \otimes_K \mathbf{C}_K \oplus H^0(A, \Omega^1_{A/K}) \otimes_K \mathbf{C}_K(-1)$$

In the same paper, Tate asked whether one could obtain similar decompositions for smooth proper varieties over $\mathcal{O}_K$ and K, or even for suitable rigid analytic spaces. The first of these questions was answered by Faltings, whose result in the algebraic setting is as follows.

THEOREM 1.4 (Faltings). *Let X be any smooth, proper variety over a p-adic field K. Then there is a canonical isomorphism*

$$H^n_{\acute{e}t}(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_K \cong \bigoplus_{p+q=n} H^q(X, \Omega^p_X) \otimes_K \mathbf{C}_K(-p) \tag{2.1}$$

which is functorial and Galois equivariant.

Let us briefly discuss one idea in Faltings' proof, since it will also appear in Scholze's work. Rather than trying to compare the two cohomology theories directly, Faltings introduces a third theory $\mathcal{H}^*(X)$ related to both of them. It turns out that one can understand this intermediate theory quite naturally in terms of a new site on a rigid space, the pro-étale site. This is one of the ideas that allows Scholze to treat the rigid analytic case.

THEOREM 1.5 (Scholze). *Let K be a p-adic field, and X a smooth proper rigid analytic space over K. Then:*

- *$H^*_{\acute{e}t}(X_{\bar{K}}, \mathbf{Q}_p)$ is finite-dimensional as a $\mathbf{Q}_p$-vector space, equal to 0 if $* > 2 \dim X$.*
- *$H^*_{\acute{e}t}(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_K$ admits the same decomposition (2.1) as in Faltings' theorem.*
- *(2.1) is the 0-th graded piece of a canonical isomorphism*

  $$H^*_{\acute{e}t}(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \cong H^*_{\mathrm{dR}}(X) \otimes_K B_{\mathrm{dR}}$$

  which is again functorial and Galois equivariant.
- *The Hodge-de Rham spectral sequence degenerates.*

Using suitable GAGA theorems, one recovers Faltings' decomposition from Theorem 1.5. Before getting into the details, let us see roughly how the proof works. The point is that we do not compare étale cohomology and de Rham cohomology all at once, but pass through several intermediate cohomology groups.

(i) Define the pro-étale site $X_{\mathrm{pro\acute{e}t}}$ and construct its basis of affinoid perfectoid objects. These objects have the almost acyclicity properties that will replace local contractibility in the argument.

(ii) Define sheaves of topological rings $\hat{\mathcal{O}}_X$ and $\mathbf{B}_{\mathrm{dR}}$ on $X_{\mathrm{pro\acute{e}t}}$. Then construct isomorphisms $H^*_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_K \cong H^*_{\mathrm{pro\acute{e}t}}(X_{\bar{K}}, \hat{\mathcal{O}}_X)$. This "pro-étale cohomology"

serves the role of Faltings' intermediate cohomology theory $\mathcal{H}^*(X)$. In the same way, one establishes the comparison $H^*_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\text{dR}} \cong H^*_{\text{proét}}(X_{\bar{K}}, \mathbf{B}_{\text{dR}})$.

(iii) Use a Poincaré lemma to express the de Rham period sheaf in terms of differentials. More precisely, prove the exactness of the sequence

$$0 \to \mathbf{B}_{\text{dR}} \to \mathcal{O}\mathbf{B}_{\text{dR}} \to \mathcal{O}\mathbf{B}_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega^1_X \to \cdots \to \mathcal{O}\mathbf{B}_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega^{\dim X}_X \to 0$$

where $\mathcal{O}\mathbf{B}_{\text{dR}}$ is a certain sheaf of $\mathbf{B}_{\text{dR}}$-algebras on $X_{\text{proét}}$ which (unlike $\mathbf{B}_{\text{dR}}$) is also a sheaf of $\mathcal{O}_X$-algebras. This implies an isomorphism

$$H^*_{\text{proét}}(X_{\bar{K}}, \mathbf{B}_{\text{dR}}) \cong \mathbf{H}^*_{\text{proét}}(X_{\bar{K}}, \mathcal{O}\mathbf{B}_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega^\bullet_X).$$

(iv) There is a natural projection $v\colon X_{\bar{K},\text{proét}} \to X_{\bar{K}}$. Prove $v_*\mathcal{O}\mathbf{B}_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega^\bullet_X \cong Rv_*\mathcal{O}\mathbf{B}_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega^\bullet_X$. This is called the *geometric acyclicity* of $\mathcal{O}\mathbf{B}_{\text{dR}}$. Combining this with the previous two steps, we obtain an isomorphism $H^*_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p) \otimes B_{\text{dR}} \cong \mathbf{H}^*(X_{\bar{K}}, v_*\mathcal{O}\mathbf{B}_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega^\bullet_X)$.

(v) Finally, identify the right-hand side with de Rham cohomology after extension of scalars to the period ring.

3. Galois Representations

There is another way to look at Scholze's theorem that will be useful to us. The étale cohomology groups on the left-hand side of the comparison are p-adic Galois representations, so the theorem tells us something about which representations come from geometry. Fontaine's idea was to study a representation by tensoring it with a suitable period ring. In our case, we will be particularly interested in the representations he calls *de Rham*.

One reason de Rham representations are so interesting is that they admit a monodromy theorem in the spirit of Grothendieck's theorem for ℓ-adic representations. More precisely, Berger showed that every de Rham representation is potentially semistable.

Scholze's theorem says, in particular, that the étale cohomology groups of smooth proper rigid spaces are de Rham representations of the Galois group of the base field. Thus the rigid analytic setting gives us many more geometric examples to study than the algebraic setting alone. We will return to these representations at the end of the thesis.

4. Structure of This Thesis

We start in Chapter 2 by comparing the basic theory of ℓ-adic and p-adic representations. This will motivate the definitions of the period rings B_{HT} and B_{dR}. In Chapter 3 we recall the construction of rigid spaces and discuss why it is convenient to work with Huber's adic spaces instead. The reason for doing this will become apparent in Chapter 4, when we introduce perfectoid spaces, our main geometric tool in the proof of Theorem 1.5.

The actual comparison takes place in Chapter 5. We first construct the pro-étale site and see how the perfectoid spaces of Chapter 4 enter the picture. This takes care of step (i) in the outline above. We can then carry out the remaining steps to obtain Theorem 5.56. Chapter 6 is devoted to some applications: we return to the question of geometric de Rham representations, and see what the decomposition tells us about Hodge numbers, subquotients, and algebraic cycle classes. We end by discussing some

results of Bhatt-Morrow-Scholze on what happens when one works with integral, rather than rational, cohomology.

CHAPTER 2

p-adic Galois Representations

It is perhaps easiest to see what makes p-adic representations of p-adic fields interesting by first looking at ℓ-adic representations, where $\ell \neq p$. Much of the theory is considerably simpler in this case. We will see that some very useful results about these representations fail when $\ell = p$, and this will give us a good reason to introduce Fontaine's period rings. We mainly follow [**FO**] in this chapter.

1. ℓ-adic Representations

1.1. Basic Theory. Throughout this section, K will be a finite extension of $\mathbf{Q}_p$, ℓ will be a prime distinct from p, and we write $G_K := \mathrm{Gal}(\bar{K}/K)$.

Definition 2.1. An ℓ-adic representation of G_K is a pair (ρ, V), where V is a finite-dimensional $\mathbf{Q}_\ell$-vector space and $\rho\colon G_K \to \mathrm{GL}(V)$ is a continuous homomorphism.

It will often be convenient to work integrally, rather than over $\mathbf{Q}_\ell$. This leads us to consider representations on finitely generated $\mathbf{Z}_\ell$-modules.

Definition 2.2. A $\mathbf{Z}_\ell$-representation of G_K is a pair (ρ, M), where M is a finitely generated $\mathbf{Z}_\ell$-module, and $\rho\colon G_K \to \mathrm{GL}(M)$ is a continuous homomorphism.

To relate these two notions, recall that a lattice in a $\mathbf{Q}_\ell$-vector space W is a finitely generated free $\mathbf{Z}_\ell$-submodule L of W such that $L \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell = W$.

Lemma 2.3. *For any ℓ-adic representation V of G_K, there is a lattice T in V which is stable under the G_K-action, and is thus a free $\mathbf{Z}_\ell$-representation of G_K.*

Proof. Choose any lattice $T_0 \subset V$. Its stabilizer is open in G_K: after choosing a basis of T_0, this follows from the continuity of ρ and the fact that $\mathrm{GL}_d(\mathbf{Z}_\ell)$ is open in $\mathrm{GL}_d(\mathbf{Q}_\ell)$. Since G_K is compact, this stabilizer has finite index. Thus there are only finitely many distinct lattices of the form gT_0, and we may set

$$T = \sum_{g \in G_K} gT_0.$$

This is a finite sum of finitely generated $\mathbf{Z}_\ell$-modules. It is torsion-free, since it is contained in V, and it spans V, since it contains T_0. Hence it is a lattice. Finally, left multiplication permutes the summands, so T is G_K-stable. □

The usefulness of a stable lattice becomes apparent when we study the action of inertia. Let k denote the residue field of our local field K (a perfect field of characteristic p), and write I_K and P_K for the inertia and wild inertia subgroups of G_K, respectively.

There are short exact sequences

$$1 \to I_K \to G_K \to G_k \to 1$$

$$1 \to P_K \to I_K \to I_K/P_K \to 1$$

Under the isomorphism

$$I_K/P_K \cong \widehat{\mathbf{Z}}'(1) = \prod_{\ell' \neq p} \mathbf{Z}_{\ell'}(1) = \mathbf{Z}_\ell(1) \times \prod_{\ell' \neq \ell, p} \mathbf{Z}_{\ell'}(1)$$

, we let $P_{K,\ell}$ be the preimage of the product $\prod_{\ell' \neq \ell,p} \mathbf{Z}_{\ell'}(1)$ in I_K and set $G_{K,\ell} := G_K/P_{K,\ell}$. Then by construction we have $I_K/P_{K,\ell} \cong \mathbf{Z}_\ell(1)$, and therefore a short exact sequence

$$1 \to \mathbf{Z}_\ell(1) \to G_{K,\ell} \to G_k \to 1$$

To see what the tame quotient tells us about representations, let V be any ℓ-adic representation of G_K, and let T be a $\mathbf{Z}_\ell$-lattice stable under the G_K-action, guaranteed by Lemma 2.3. Then the corresponding map $\rho\colon G_K \to \mathrm{Aut}_{\mathbf{Q}_\ell}(V)$ factors through $\mathrm{Aut}_{\mathbf{Z}_\ell}(T)$ and in fact $\rho(G_K)$ is a closed subgroup of $\mathrm{Aut}_{\mathbf{Z}_\ell}(T)$. Let d be the dimension of V, so we have isomorphisms $\mathrm{Aut}_{\mathbf{Q}_\ell}(V) \cong \mathrm{GL}_d(\mathbf{Q}_\ell)$ and $\mathrm{Aut}_{\mathbf{Z}_\ell}(T) \cong \mathrm{GL}_d(\mathbf{Z}_\ell)$.

The key point is that the reduction map has a pro-ℓ kernel. Consider the short exact sequence

$$1 \to N_1 \to \mathrm{GL}_d(\mathbf{Z}_\ell) \to \mathrm{GL}_d(\mathbf{F}_\ell) \to 1$$

where N_1 is the kernel of the reduction-mod-ℓ map. Let N_n be the subgroup of N_1 of matrices congruent to 1 mod ℓ^n for $n \geqslant 1$. Then we have $N_1 \cong \varprojlim N_1/N_n$, so N_1 is a pro-ℓ-group.

By the exact sequence

$$1 \to P_K \to P_{K,\ell} \to \prod_{\ell' \neq p,\ell} \mathbf{Z}_{\ell'}(1) \to 1$$

we see that since P_K is a pro-p group, $P_{K,\ell}$ is the inverse limit of groups with prime-to-ℓ orders, and thus $\rho(P_{K,\ell}) \cap N_1 = \{1\}$. Thus the image $\rho(P_{K,\ell}) \hookrightarrow \mathrm{GL}_d(\mathbf{F}_\ell)$ is finite.

Definition 2.4. Let $\rho\colon G_K \to \mathrm{Aut}_{\mathbf{Q}_\ell}(V)$ be an ℓ-adic representation of G_K.

(i) V is unramified, or has good reduction, if I_K acts trivially. Equivalently, $\rho(I_K) = \{1\}$.
(ii) V has potentially good reduction if $\rho(I_K)$ has finite image. Equivalently, there is a finite extension K'/K such that V, when viewed as an ℓ-adic representation of $G_{K'}$, has good reduction.
(iii) V is semistable if I_K acts unipotently.
(iv) V is potentially semistable if there is a finite extension K'/K such that V, when viewed as an ℓ-adic representation of $G_{K'}$, is semistable. Equivalently, there is an open subgroup of I_K which acts unipotently.

Theorem 2.5 (Grothendieck's ℓ-adic Monodromy Theorem). *Assume that the group*

$$\mu_{\ell^\infty}(K(\mu_\ell)) := \{\epsilon \in K(\mu_\ell) : \epsilon^{\ell^n} = 1 \text{ for some } n\}$$

is finite. Then any ℓ-adic representation of G_K is potentially semistable.

Proof. By replacing K by a suitable finite extension, we may assume that $P_{K,\ell}$ acts trivially. This means that ρ factors through $G_{K,\ell}$, and let $\bar{\rho}\colon G_{K,\ell} \to \mathrm{Aut}_{\mathbf{Q}_\ell}(V)$ be the corresponding factorization. Consider the sequence

$$1 \to \mathbf{Z}_\ell(1) \to G_{K,\ell} \to G_k \to 1$$

If t is a topological generator of $\mathbf{Z}_\ell(1)$, then $\bar{\rho}(t) \in \mathrm{Aut}_{\mathbf{Q}_\ell}(V)$. Choose a finite extension E of $\mathbf{Q}_\ell$ such that the characteristic polynomial of $\bar{\rho}(t)$ splits in E. Setting $V' = E \otimes_{\mathbf{Q}_\ell} V$,

we see that V' is an E-representation of $G_{K,\ell}$ via the action $g(\lambda \otimes v) = \lambda \otimes g(v)$. Let a be an eigenvalue of $\bar{\rho}(t)$ and $v \in V'$ the corresponding eigenvector.

Let $\chi_\ell \colon G_{K,\ell} \to \mathbf{Z}_\ell^\times$ be the cyclotomic character. If $g \in G_{K,\ell}$, then $gtg^{-1} = t^{\chi_\ell(g)}$. This implies

$$\bar{\rho}(gtg^{-1})(v) = \bar{\rho}(t^{\chi_\ell(g)})(v) = a^{\chi_\ell(g)}v$$

and therefore $\bar{\rho}(t)(g^{-1}(v)) = a^{\chi_\ell(g)}g^{-1}(v)$. Thus, if a is an eigenvalue of $\bar{\rho}(t)$, then for all $n \in \mathbf{Z}$ such that there exists $g \in G_{K,\ell}$ with $\chi_\ell(g) = n$, a^n is also an eigenvalue of $\bar{\rho}(t)$. Because of the finiteness condition imposed on $\mu_{\ell^\infty}(K(\mu_\ell))$, we see that the image of χ_ℓ in $\mathbf{Z}_\ell^\times$ is open, hence infinite, so there are infinitely many such n's. Thus the eigenvalue a must be a root of unity, and there exists some $N \geqslant 1$ for which t^N acts unipotently. Then the closure of the subgroup generated by t^N acts unipotently, and is an open subgroup of $\mathbf{Z}_\ell(1)$. Since I_K surjects onto $\mathbf{Z}_\ell(1)$, its inverse image is an open subgroup of inertia acting unipotently. This is precisely the potential semistability we sought. □

Once we pass to a finite extension, all that remains of the inertia action is unipotent. One can use this to give a very convenient description of representations, which we discuss next.

1.2. Applications. To see how the monodromy theorem can simplify the data of an ℓ-adic representation, let us first look at the Weil group. We will then give an application to the étale cohomology of varieties.

First, we recall the definition of the Weil group. Let K be a finite extension of $\mathbf{Q}_p$, with residue field k. Since $G_k \cong \hat{\mathbf{Z}}$, we have a short exact sequence

$$1 \to I_K \to G_K \to \hat{\mathbf{Z}} \to 1$$

Definition 2.6. The Weil group W_K is defined as the pullback

$$\begin{array}{ccc} W_K & \longrightarrow & \mathbf{Z} \\ \downarrow & \lrcorner & \downarrow \\ G_K & \longrightarrow & \hat{\mathbf{Z}} \end{array}$$

Let us put the coefficient fields into the definition carefully. Fix a finite extension $E/\mathbf{Q}_\ell$, and choose a lift $\Phi \in W_K$ of geometric Frobenius. Write $q = |k|$ and $d \colon W_K \to \mathbf{Z}$ for the map taking Φ to 1. We put $\nu(w) = q^{-d(w)}$. The topology on W_K is the one for which I_K is open with its profinite topology.

Definition 2.7. A Weil-Deligne representation over E is a triple (r, V, N), where V is a finite-dimensional E-vector space, $r \colon W_K \to \mathrm{GL}_E(V)$ is a homomorphism with open kernel on inertia, and $N \in \mathrm{End}_E(V)$ is nilpotent, with

$$r(w)Nr(w)^{-1} = \nu(w)N.$$

It is Frobenius semisimple if $r(\Phi)$ is semisimple. We call it *bounded at Frobenius* if every eigenvalue of $r(\Phi)$ is an ℓ-adic unit.

Since inertia is compact, the open-kernel condition says exactly that $r(I_K)$ is finite. Notice that Frobenius semisimplicity is not part of the definition of a Weil-Deligne representation.

THEOREM 2.8 (Deligne). *After fixing a generator of $\mathbf{Z}_\ell(1)$ and the lift Φ, continuous finite-dimensional E-representations of G_K are equivalent to Weil-Deligne representations over E that are bounded at Frobenius.*

PROOF. Let $t_\ell \colon I_K \to \mathbf{Z}_\ell$ be the tame character in the chosen basis. The monodromy theorem gives an open normal subgroup $I' \subset I_K$ on which a given representation ρ is unipotent and factors through the ℓ-primary tame quotient. The logarithm of a unipotent matrix is a finite sum. Taking logarithms therefore turns this action into an additive continuous map on an open subgroup of $\mathbf{Z}_\ell$. There is a unique nilpotent operator N such that

$$\rho(\sigma) = \exp(t_\ell(\sigma)N), \ \sigma \in I'.$$

Here the exponential is also a finite sum. The uniqueness follows by evaluating at an element whose tame character is nonzero. Conjugation of inertia gives

$$\rho(w)N\rho(w)^{-1} = \nu(w)N.$$

Every $w \in W_K$ has a unique expression $w = \sigma\Phi^{d(w)}$. Extend t_ℓ to W_K by $t_\ell(w) = t_\ell(\sigma)$. It satisfies

$$t_\ell(w_1w_2) = t_\ell(w_1) + \nu(w_1)t_\ell(w_2).$$

Consequently

$$r(w) = \exp(-t_\ell(w)N)\rho(w)$$

is a homomorphism: the displayed cocycle identity is exactly what is needed when multiplying the two exponentials. On I' it is trivial, so its inertia image is finite. Also $r(\Phi) = \rho(\Phi)$. A stable lattice, constructed just as in Lemma 2.3, shows that this matrix and its inverse are integral after a change of basis. Its eigenvalues are therefore units.

Conversely, start with (r, V, N) as in the statement and set

$$\rho(w) = \exp(t_\ell(w)N)r(w).$$

The same calculation proves multiplicativity. Its restriction to inertia is continuous, and its image is bounded: $r(I_K)$ is finite, while the exponential is a polynomial in the bounded variable $t_\ell(\sigma) \in \mathbf{Z}_\ell$. The powers of $r(\Phi)$ are bounded in both directions as well. To see the last assertion without diagonalizing, its characteristic polynomial has integral coefficients and unit constant term. Cayley–Hamilton then puts all integral powers in the module generated by finitely many powers of the matrix. Hence the closure of the cyclic group it generates is compact.

The chosen Φ gives a splitting $G_K = I_K \rtimes \widehat{\mathbf{Z}}$. Its action on V extends from $\mathbf{Z}$ to $\widehat{\mathbf{Z}}$ because the closure of its powers is profinite. The conjugation relation with inertia extends by continuity, giving a continuous representation of G_K. The two constructions are inverse. A map intertwines one pair of actions if and only if it intertwines their logarithms and finite-inertia parts, so they are inverse on morphisms also. □

This explains the role of the nilpotent operator: it records the part of inertia that survives after passing to a finite extension. Complex Weil-Deligne representations also occur in the local Langlands correspondence, but one must then change coefficients and retain the appropriate Frobenius conditions when comparing with Galois representations. The local Langlands correspondence for GL_n was proved by Harris-Taylor, Henniart, and Scholze [**HT01**, **Hen00**, **Sch13a**]. We will not need it below.

Alternatively, the monodromy theorem has more concrete geometric applications. For instance:

THEOREM 2.9. *Let X be a smooth proper variety over a p-adic field K. Then the étale cohomology groups*

$$H^i_{\acute{e}t}(X_{\bar{K}}, \mathbf{Q}_\ell)$$

are potentially semistable ℓ-adic representations of G_K.

PROOF. The finiteness theorem for étale cohomology gives a finite-dimensional $\mathbf{Q}_\ell$-vector space, and its Galois action is continuous, as can be seen from the finite-coefficient cohomology groups. We can therefore apply Theorem 2.5 directly. The finiteness condition on roots of unity holds because K is a finite extension of $\mathbf{Q}_p$ and $\ell \neq p$. □

The monodromy theorem also enters other fundamental results in arithmetic geometry, among them the Néron-Ogg-Shafarevich criterion and Deligne's proof of the Weil conjectures.

Given the usefulness of the monodromy theorem in arithmetic geometry, it would be desirable to have an analogous result for p-adic representations.

2. p-adic Representations

To see why the same statement cannot hold when $\ell = p$, let us go back to the proof of the monodromy theorem. The Galois group is profinite in both cases. When $\ell \neq p$, however, wild inertia is pro-p, whereas a sufficiently small congruence subgroup of $\mathrm{GL}_d(\mathbf{Z}_\ell)$ is pro-ℓ. Their common image must be trivial, so wild inertia has finite image. For p-adic representations both groups can be pro-p, and this argument gives no restriction.

There is an especially simple example. On the one-dimensional representation $\mathbf{Q}_p(1)$, inertia acts through the cyclotomic character, whose image is infinite on every open subgroup of inertia. A unipotent operator in dimension one is the identity. Thus $\mathbf{Q}_p(1)$ does not become unipotent on inertia after any finite extension. This is why the p-adic monodromy theorem will require a different notion of semistability.

We are thus led to look for a different notion of semistability and to single out a class of p-adic representations for which a monodromy theorem holds. At the very least, this class should contain the geometric representations in the spirit of Theorem 2.9.

This was part of the motivation for the theory developed by Tate in the late 1960s and Fontaine in the early 1980s. As a first step, let us look again at what Tate's decomposition says about a representation coming from geometry.

3. Hodge-Tate Representations

Recall Tate's theorem from the introduction [**Tat67**]:

THEOREM 2.10. *Let K be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_K$. Let A be an abelian variety over K with good reduction. Then there is a decomposition*

$$H^1_{\acute{e}t}(A_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_K \cong H^1(A, \mathcal{O}_A) \otimes_K \mathbf{C}_K \oplus H^0(A, \Omega^1_A) \otimes_K \mathbf{C}_K(-1)$$

There is something very striking about this decomposition from the point of view of Galois representations. The representation $H^1_{\text{ét}}(A_{\bar{K}}, \mathbf{Q}_p)$ may be rather complicated, but after tensoring with $\mathbf{C}_K$ it is built out of just the usual Galois action on $\mathbf{C}_K$ and a twist of

this action. We can use this behavior after extending scalars as a way of distinguishing a class of p-adic representations.

A convenient way of doing this, due to Fontaine, is to put all the Tate twists of $\mathbf{C}_K$ into a single graded ring. The dimension of the space of invariant vectors after tensoring with this ring will give us the desired criterion. We will use essentially the same idea when we discuss the de Rham period ring.

DEFINITION 2.11. The ring of Hodge-Tate periods, denoted B_{HT}, is defined as

$$B_{\mathrm{HT}} = \bigoplus_{i \in \mathbf{Z}} \mathbf{C}_K(i) \cong \mathbf{C}_K[t, t^{-1}]$$

The Galois action in this presentation is $g(t) = \chi_p(g)t$. We will use one arithmetic input about $\mathbf{C}_K$, due to Tate:

THEOREM 2.12 (Tate). *We have $\mathbf{C}_K^{G_K} = K$, and $(\mathbf{C}_K(n))^{G_K} = 0$ for every nonzero integer n.*

This is the part of Tate–Sen theory needed for our discussion of period rings; see [**FO**, Corollary 3.56]. Its proof uses normalized traces in the cyclotomic tower. We use this theorem as an arithmetic input, and spell out its algebraic consequences below.

Tate's theorem has a few useful consequences for this ring. In particular, it tells us about its Galois invariants and about the elements that transform by a character.

PROPOSITION 2.13. *The ring B_{HT} is $(\mathbf{Q}_p, G_K)$-regular, which means:*

(i) *B_{HT} is a domain.*
(ii) *$(\operatorname{Frac} B_{\mathrm{HT}})^{G_K} = B_{\mathrm{HT}}^{G_K} = K$.*
(iii) *Suppose b is a nonzero element of B_{HT} such that for all $g \in G_K$, we have that $g(b) \in \mathbf{Q}_p b$. Then b is a unit.*

PROOF. The Laurent polynomial description shows that B_{HT} is a domain. Embed its fraction field in $\mathbf{C}_K((t))$ by expanding at $t = 0$. This embedding respects the Galois action. If a Laurent series $\sum_i a_i t^i$ is invariant, each term $a_i t^i$ is invariant separately. Theorem 2.12 says that all terms with $i \neq 0$ vanish and that $a_0 \in K$. This proves the assertion about invariants.

For the last assertion, write $b = \sum_i a_i t^i$. If two coefficients a_i, a_j are nonzero, the hypothesis gives

$$g(a_i/a_j) = \chi_p(g)^{j-i}(a_i/a_j).$$

Thus $(a_i/a_j)t^{i-j}$ is a nonzero invariant in $\mathbf{C}_K(i-j)$. The same theorem forces $i = j$. Hence b has just one nonzero term, and every nonzero monomial in a Laurent polynomial ring over a field is a unit. □

If V is any p-adic representation, define $\mathbf{D}_{\mathrm{HT}}(V) := (B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V)^{G_K}$. There is a canonical map $\alpha_V : B_{\mathrm{HT}} \otimes_K \mathbf{D}_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbf{Q}_p} V$. Let us justify the injectivity assertion that makes this construction useful.

LEMMA 2.14. *Let B be a $(\mathbf{Q}_p, G_K)$-regular ring, with $B^{G_K} = K$, and let V be a finite-dimensional $\mathbf{Q}_p$-representation. Then*

$$B \otimes_K (B \otimes_{\mathbf{Q}_p} V)^{G_K} \longrightarrow B \otimes_{\mathbf{Q}_p} V$$

is injective. The dimension of the space of invariants is at most $\dim_{\mathbf{Q}_p} V$, and equality holds if and only if this map is an isomorphism.

PROOF. Put $F = \mathrm{Frac}(B)$, so $F^{G_K} = K$. Suppose G_K-invariant vectors $v_1, \ldots, v_r$ are independent over K but dependent over F. Choose a dependence of minimal length and divide by its last nonzero coefficient, obtaining $\sum_{j<r} a_j v_j + v_r = 0$. Applying g and subtracting gives $\sum_{j<r}(g(a_j) - a_j)v_j = 0$. Minimality makes all these coefficients zero. Thus each $a_j \in K$, contradicting the original independence. This proves injectivity and the dimension bound after passing to F, hence also over B.

If equality holds, choose bases and let D be the matrix of this map. Its determinant b is nonzero, and invariance of its columns gives $g(b) = \det(\rho(g))^{-1} b$. Regularity makes b a unit of B. The adjugate formula now gives an inverse to D over B. The converse follows by comparing ranks after tensoring with F. □

DEFINITION 2.15. Let V be any p-adic representation of G_K. We say that V is B_{HT}-admissible, or Hodge-Tate, if the map α_V is an isomorphism. Thus V is Hodge-Tate if and only if $\dim_K \mathbf{D}_{\mathrm{HT}}(V) = \dim_{\mathbf{Q}_p} V$.

The grading gives us a little more information than just the total dimension of the space of invariants: we can count the invariant periods of each degree separately.

DEFINITION 2.16. The Hodge-Tate numbers of a p-adic representation V of G_K are the integers

$$h_i := \dim_K(\mathbf{C}_K(-i) \otimes_{\mathbf{Q}_p} V)^{G_K}$$

for $i \in \mathbf{Z}$.

EXAMPLE 2.17. Let E be an elliptic curve over K. Then the rational Tate module $V_p(E)$ is a 2-dimensional Hodge-Tate representation, and we have

$$\dim_K(\mathbf{C}_K \otimes_{\mathbf{Q}_p} V_p(E))^{G_K} = \dim_K(\mathbf{C}_K(-1) \otimes_{\mathbf{Q}_p} V_p(E))^{G_K} = 1$$

So the Hodge-Tate numbers are $h_0 = h_1 = 1$.

4. de Rham Representations

So far, our period ring has consisted of the various twists of $\mathbf{C}_K$, each sitting in its own degree. But recall that de Rham cohomology naturally comes with a filtration. What we would like is a filtered ring whose associated graded ring is B_{HT}. To get such a ring, we will first construct a map from a larger ring onto $\mathbf{C}_K$.

4.1. The map θ. Let K be a finite extension of $\mathbf{Q}_p$, put $C = \mathbf{C}_K$, and write $\mathcal{O}_C$ for its ring of integers. We first give a concrete meaning to the tilt used in this construction:

$$R = \mathcal{O}_C^\flat := \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p.$$

Thus an element $x \in R$ is a compatible sequence $(x_0, x_1, \ldots)$ of residue classes. If $\widetilde{x}_n \in \mathcal{O}_C$ is any lift of x_n, define

$$x^\sharp = \lim_{n \to \infty} \widetilde{x}_n^{p^n}.$$

The elementary congruence $a \equiv b \pmod{p}$ implies $a^{p^m} \equiv b^{p^m} \pmod{p^{m+1}}$, by induction using the binomial theorem. It shows both convergence and independence of the chosen lifts. This map is multiplicative; it is generally not additive. The ring R is a perfect valuation ring, with valuation $v_R(x) = v_C(x^\sharp)$ and $v_C(p) = 1$.

Every Witt vector has a unique convergent Teichmüller expansion $a = \sum_{n\geqslant 0} p^n[x_n]$. Fontaine's map is

$$\theta\colon W(R) \longrightarrow \mathcal{O}_C,\ \theta\left(\sum_{n\geqslant 0} p^n[x_n]\right) = \sum_{n\geqslant 0} p^n x_n^\sharp.$$

The Witt addition and multiplication polynomials show that this is a ring map. One way to check this is to reduce modulo p^m: choose a sufficiently far coordinate in each compatible sequence and apply the ghost-component formulas there. The congruence just proved makes the resulting formula independent of the lifts, and those formulas respect sums and products. Passing to all m gives the asserted ring map. This also makes its G_K-equivariance clear. We use the basic construction of Witt vectors here; the same construction is described in [**FO**, Section 5.2.1].

Proposition 2.18. *The map θ is surjective.*

Proof. Let $a \in \mathcal{O}_C$. Since C is algebraically closed, choose $a_0 = a, a_1, a_2, \ldots$ with $a_{n+1}^p = a_n$. All these elements are integral. Their reductions define $x \in R$ with $x^\sharp = a$, so $\theta([x]) = a$. □

Choose compatible p-power roots of $-p$, and let $\varpi \in R$ be the resulting element, so $\varpi^\sharp = -p$. Set $\xi = [\varpi] + p$. Notice that the equality $\varpi^\sharp = -p$ concerns the lift to $\mathcal{O}_C$, not the first residue-class coordinate, which is zero.

Proposition 2.19. *The kernel of θ is generated by ξ, and $\bigcap_{n\geqslant 0}(\xi)^n = 0$ in $W(R)$.*

Proof. Certainly $\theta(\xi) = 0$. Take $a \in \ker\theta$ and write $a = [x] + pa_1$. Then $x^\sharp = -p\theta(a_1)$, so $v_R(x) \geqslant 1 = v_R(\varpi)$. The valuation-ring property gives $x = \varpi y$ for some $y \in R$. Hence $a - \xi[y]$ is divisible by p in $W(R)$. Its quotient still lies in $\ker\theta$, since $\mathcal{O}_C$ has no p-torsion. Repeating this construction and using p-adic completeness gives

$$a = \xi\big([y_0] + p[y_1] + p^2[y_2] + \cdots\big).$$

Thus $\ker\theta = (\xi)$.

The ring $W(R)$ is a domain: for two nonzero Witt vectors, divide out their largest powers of p and reduce the remaining product modulo p in the domain R. Also every ideal $(\xi)^n$ is p-saturated. Indeed, if $pb \in (\xi)^n$, applying θ first shows $b \in (\xi)$; cancelling ξ and arguing inductively proves $b \in (\xi)^n$.

Now let a belong to every $(\xi)^n$. Modulo p its image belongs to every $(\varpi)^n$, whose intersection is zero by the valuation. Hence $a = pb$. Saturation puts b in the same intersection. Repeating this shows that a is divisible by every power of p, and p-adic separatedness gives $a = 0$. □

We have now constructed a ring surjecting onto $\mathcal{O}_C$, with a principal kernel. The idea is to keep track not just of the quotient by this kernel, but of the quotients by all its powers. After inverting p, we complete with respect to this ideal.

Definition 2.20. We let B_{dR}^+ denote the $(\ker\theta)$-adic completion of $W(R)[p^{-1}]$. We let B_{dR} denote the fraction field of B_{dR}^+.

There is a small point in this definition that we should justify: the completed ring is in fact a discrete valuation ring.

LEMMA 2.21. *The ring B_{dR}^+ is a complete discrete valuation ring with uniformizer ξ and residue field C.*

PROOF. Put $S = W(R)[p^{-1}]$. The preceding proposition gives $S/(\xi) = C$. Its p-saturation assertion also shows that S is ξ-adically separated: clearing a power of p reduces the intersection of its powers of ξ to the corresponding intersection in $W(R)$. Since ξ is a nonzerodivisor, multiplication by ξ^n identifies $S/(\xi)$ with $(\xi)^n/(\xi)^{n+1}$. The transition maps of these quotient systems are surjective, so these statements persist in the inverse limit defining B_{dR}^+. Its reduction modulo ξ is therefore C, and its kernel is ξB_{dR}^+.

An element whose reduction is nonzero is a unit: lift the inverse of its reduction and invert the resulting element $1 + \xi z$ by a geometric series. Every nonzero element has a first nonzero graded piece, by separatedness, and is consequently ξ^n times a unit. Thus every nonzero ideal is generated by the element in it of smallest valuation. This proves the assertion. □

The Galois action preserves $\ker\theta$, so it extends to B_{dR}^+ and its fraction field. For continuity, give each $(W(R)/(\xi^n))[1/p]$ its quotient Banach topology, obtained from the p-adic topology on $W(R)/(\xi^n)$ before inverting p, and give B_{dR}^+ the resulting inverse-limit topology. This is the natural topology used for continuous Galois actions. The purely discrete valuation topology alone does not describe this continuity.

There is also a canonical copy of $\bar{K}$ in B_{dR}^+ compatible with its given embedding into C. Indeed, a simple algebraic root in C lifts uniquely by Hensel's lemma. Applying this to finite extensions of $\mathbf{Q}_p$ and using uniqueness makes the lifts compatible. In particular, $K \subset B_{\mathrm{dR}}^+$ is fixed by G_K.

The Galois action on the period ring takes care of one side of our desired comparison. On the de Rham side, however, we also have the Hodge filtration to account for. Fortunately, the discrete valuation on B_{dR}^+ gives us a very natural filtration.

DEFINITION 2.22. For $i \in \mathbf{Z}$, let $\mathrm{Fil}^i B_{\mathrm{dR}}$ be the free B_{dR}^+-module generated by ξ^i. Then there is a decreasing, exhaustive, and separated filtration

$$\cdots \supset \mathrm{Fil}^i B_{\mathrm{dR}} \supset \mathrm{Fil}^{i+1} B_{\mathrm{dR}} \supset \cdots$$

Choose compatible primitive roots $\zeta_{p^n} \in \mathcal{O}_C$ with $\zeta_{p^{n+1}}^p = \zeta_{p^n}$, and let $\epsilon \in R$ be the sequence of reductions of $(1, \zeta_p, \zeta_{p^2}, \ldots)$. Thus $\epsilon^\sharp = 1$ and $(\epsilon^{1/p})^\sharp = \zeta_p$. These equalities refer to the lifts in $\mathcal{O}_C$, rather than to the coordinates modulo p. Then we may define $\pi := [\epsilon] - 1 \in W(R)$, and π satisfies $\theta(\pi) = 0$. Therefore $\pi \in \ker\theta$, and so $[\epsilon] - 1 \in \mathrm{Fil}^1 B_{\mathrm{dR}}$. Then note that

$$(-1)^{n+1}\frac{([\epsilon]-1)^n}{n} \in W(R)[p^{-1}]\xi^n$$

and therefore $\log[\epsilon] \in B_{\mathrm{dR}}^+$. We let $t := \log[\epsilon]$.

PROPOSITION 2.23. *The element t generates the maximal ideal of B_{dR}^+. Equivalently, $t \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ and $t \notin \mathrm{Fil}^2 B_{\mathrm{dR}}$.*

PROOF. Put $u = [\epsilon^{1/p}]$ and $\eta = 1 + u + \cdots + u^{p-1}$. Then

$$[\epsilon] - 1 = (u-1)\eta,\ \theta(u-1) = \zeta_p - 1 \neq 0,\ \theta(\eta) = 0.$$

We claim that η is another generator of $\ker\theta$. Modulo p, it is $(\epsilon^{1/p}-1)^{p-1}$. The valuation formula

$$v_C(\zeta_{p^m}-1)=\frac{1}{p^{m-1}(p-1)}$$

follows by taking the valuation of the cyclotomic identity $\Phi_{p^m}(1)=p$; all primitive roots have the same valuation. Passing to the tilt gives $v_R(\epsilon^{1/p}-1)=1/(p-1)$. Hence the reduction of η has valuation 1, just as ϖ does. Since $\eta=\xi a$ by Proposition 2.19, the reduction of a is a unit of R, and a is a unit of $W(R)$.

The element $u-1$ is a unit in B_{dR}^+ because its image under θ is nonzero. Thus $[\epsilon]-1$ has valuation exactly one in this discrete valuation ring. Finally,

$$t=([\epsilon]-1)\left(1-\frac{[\epsilon]-1}{2}+\frac{([\epsilon]-1)^2}{3}-\cdots\right),$$

and the parenthesized factor is congruent to 1 modulo ξ. It is a unit, so t is also a uniformizer. This argument works for $p=2$ as well. □

The logarithm identity also gives $g(t)=\chi_p(g)t$: apply it to $g([\epsilon])=[\epsilon]^{\chi_p(g)}$. Thus the powers of t retain exactly the cyclotomic twists that we saw in B_{HT}.

Recall why we wanted a filtration in the first place. On passing to the associated graded, we would like to recover the twists of C that make up B_{HT}. This is indeed what happens.

Proposition 2.24. *There is a Galois-equivariant graded isomorphism* $\mathrm{gr}^\bullet B_{\mathrm{dR}}\cong B_{\mathrm{HT}}$.

Proof. Since t is a uniformizer, multiplication by t^i identifies C with $t^iB_{\mathrm{dR}}^+/t^{i+1}B_{\mathrm{dR}}^+$. Under this identification the Galois action is the usual action on C multiplied by χ_p^i. Thus the i-th graded piece is $C(i)$. Multiplication of powers of t is compatible with these identifications, so taking the direct sum over all $i\in\mathbf{Z}$ proves the assertion as an isomorphism of graded rings. □

We can now do for B_{dR} what we did for B_{HT}. First, though, we need to know which elements of B_{dR} are fixed by Galois.

Proposition 2.25. $B_{\mathrm{dR}}^{G_K}=K$.

Proof. We already have the inclusion $K\subset B_{\mathrm{dR}}^{G_K}$. Conversely, let b be a nonzero invariant, and let i be its t-adic valuation. Its leading term is a nonzero invariant in $\mathrm{gr}^i B_{\mathrm{dR}}=C(i)$, so Theorem 2.12 forces $i=0$. Its residue is therefore some $a\in C^{G_K}=K$. Subtract the canonical lift of a. If $b-a$ were nonzero, it would be an invariant of positive valuation, contradicting the same argument. Hence $b=a\in K$. □

The other conditions for regularity are automatic, since B_{dR} is a field containing K. Thus B_{dR} is $(\mathbf{Q}_p,G_K)$-regular: more explicitly,

(i) B_{dR} is a domain.
(ii) $(\mathrm{Frac}\,B_{\mathrm{dR}})^{G_K}=B_{\mathrm{dR}}^{G_K}=K$.
(iii) If b is a nonzero element of B_{dR} such that for all $g\in G_K$, we have that $g(b)\in\mathbf{Q}_pb$, then b is a unit.

Definition 2.26. Let V be a p-adic representation of G_K. We let $\mathbf{D}_{\mathrm{dR}}(V):=(V\otimes_{\mathbf{Q}_p}B_{\mathrm{dR}})^{G_K}$. By Lemma 2.14, there is a canonical injective map $\alpha_V\colon B_{\mathrm{dR}}\otimes_K\mathbf{D}_{\mathrm{dR}}(V)\to B_{\mathrm{dR}}\otimes_{\mathbf{Q}_p}V$; if this map is an isomorphism, we say that V is de Rham, or B_{dR}-admissible.

One of the main reasons for studying de Rham representations is that the representations obtained from the étale cohomology of smooth proper spaces are de Rham. This is a consequence of work of Faltings, Tsuji, and Scholze [**Fal88**, **Tsu99**, **Sch13b**]. Here is the version of the result that will concern us.

THEOREM 2.27 (Scholze). *Let X be a smooth proper adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$, where K is a finite extension of $\mathbf{Q}_p$. Then the étale cohomology groups $H^i_{\acute{e}t}(X_{\bar{K}}, \mathbf{Q}_p)$ are de Rham representations of G_K.*

Before turning to Scholze's proof, it is worth returning to our original motivation for studying de Rham representations. A theorem of Berger relates them to a p-adic analogue of the monodromy theorem [**Ber02**]. To state it, we need to be a bit more precise about semistability in the p-adic setting.

Fontaine's semistable period ring B_{st} defines

$$D_{\mathrm{st},L}(V) = (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}.$$

If L_0 is the maximal unramified subfield of a finite extension L/K, then V is semistable over L when $\dim_{L_0} D_{\mathrm{st},L}(V) = \dim_{\mathbf{Q}_p} V$, and potentially semistable when this holds for some such L. The construction of B_{st} is beyond what we need for the comparison theorem; see [**FO**, Chapter 6].

THEOREM 2.28 (Berger). *Let V be a de Rham representation of G_K, where K is a finite extension of $\mathbf{Q}_p$. Then V is potentially semistable in this sense.*

The proof, which we will not go into here, relies on work of Kedlaya, Fontaine, and many others. So far we have used geometry to produce representations satisfying certain conditions. Over $\mathbf{Q}$, the Fontaine-Mazur conjecture suggests that these conditions also go a long way towards characterizing the representations that come from geometry.

DEFINITION 2.29. An ℓ-adic representation (ρ, V) of $G_{\mathbf{Q}}$ is geometric if:

(i) V is unramified away from finitely many primes, i.e., for all but finitely many primes, $\rho(I_p)$ is trivial.
(ii) The restriction of V to a decomposition group at ℓ is de Rham.

CONJECTURE 2.30 (Fontaine-Mazur [**FM95**]). Let V be an irreducible geometric ℓ-adic representation of $G_{\mathbf{Q}}$. Then there are a smooth projective variety $X/\mathbf{Q}$ and integers $i \geqslant 0$ and r such that V is isomorphic to a subquotient of $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)(r)$.

CHAPTER 3

Adic Spaces

1. Rigid Geometry

Having discussed the Galois representations that will appear in our comparison, we turn to the geometry of the spaces themselves. We begin with rigid geometry, which Tate developed in order to make sense of the uniformization of elliptic curves over $\mathbf{C}_p$ [Tat71].

It is useful to keep in mind what happens over the complex numbers. For a scheme X of finite type over $\mathbf{C}$, we start by taking the set of complex points $X(\mathbf{C})$ with its Euclidean topology. As we know from Serre's GAGA theorems, one can recover much of the algebraic geometry of X from the associated complex analytic space.

One may hope for a similar theory for varieties defined over a p-adic field, like $\mathbf{Q}_p$. Naively passing to $\mathbf{C}$ via a map $\mathbf{Q}_p \to \mathbf{C}$ does not give the analytic theory we want, however: such an abstract field embedding does not respect the p-adic topology. In particular, complex analytification does not by itself retain the continuous p-adic Galois action that we want to study.

A first guess might be to do the same thing over $\mathbf{Q}_p$, using its p-adic topology. But this topology is very different from the Euclidean topology on $\mathbf{C}$: even a disk over $\mathbf{Q}_p$ is totally disconnected! If we want to develop a useful theory of analytic functions and gluing on such disks, we have to be more careful about which open sets and coverings we use.

Despite this, the construction of rigid spaces is quite similar to the construction of algebraic varieties. The main difference is that we will not use an ordinary topology on the set of rigid points to glue them together.

Recall that in the classical construction of varieties, one has three steps:

(i) Define affine space $\mathbf{A}^n$.
(ii) Define affine varieties as closed subsets $V(I) \subset \mathbf{A}^n$.
(iii) Define general varieties as gluings of affine varieties.

The construction of rigid spaces also follows this rough pattern, with a few differences along the way.

For the rest of this chapter, let K denote a complete field with a nontrivial, rank one non-archimedean absolute value. We do not assume that K is locally compact.

1.1. Affinoid Spaces. To find an analogue of $\mathbf{A}^n$ in rigid geometry, it is useful to keep in mind the kinds of open sets we want to consider. In the algebraic setting, the basic opens were of the form $\{x : f(x) \neq 0\}$. Here we also want to talk about sets given by inequalities, such as $\{x : |f(x)| \leqslant 1\}$, and combinations of them, which we refer to as *semialgebraic sets*.

This suggests starting with closed polydisks $\mathbf{B}_K^n$. Our first guess for the set of points of such a disk might be

$$\mathbf{B}_K^n = \{(x_1, \ldots, x_n) \in K^n : |x_i| \leqslant 1 \text{ for all } i\}$$

When K is algebraically closed, this definition makes perfect sense, but this description omits points when K is not algebraically closed. For example, the closed subset $V(x^n - a)$, where $a^{1/n} \notin K$ is empty.

We can include the missing points by borrowing a familiar construction from algebraic geometry: the closed points of a finite type K-scheme X are given by $X(\bar{K})/\operatorname{Aut}(\bar{K}/K)$. In the same spirit, we might take

$$\mathbf{B}_K^n = \mathbf{B}_K^n(\bar{K})/\operatorname{Aut}(\bar{K}/K)$$

This gives us a set of points, but it would be much nicer to describe the disk in terms of its ring of functions, as we do for affine schemes.

Recall that the closed points of a finite type affine K-scheme $X = \operatorname{Spec}(A)$ form the set $\operatorname{MSpec}(A)$ of maximal ideals of A. For the polydisk $\mathbf{B}_K^n$, the natural analogue of the coordinate ring is the ring of convergent power series

$$\begin{aligned} K\langle t_1, \dots, t_n\rangle &= \{f(t_1, \dots, t_n) \in K[[t_1, \dots, t_n]] : f(t_1, \dots, t_n) \text{ converges everywhere on } \mathbf{B}_K^n\} \\ &= \left\{ \sum_{I \in \mathbf{N}^n} a_I t^I \in K[[t_1, \dots, t_n]] : \lim_{|I| \to \infty} |a_I| = 0 \right\} \end{aligned}$$

The desired correspondence is then the assertion that the canonical map

$$\mathbf{B}_K^n(\bar{K})/\operatorname{Aut}(\bar{K}/K) \to \operatorname{MSpec} K\langle t_1, \dots, t_n\rangle$$

given by sending α to the kernel of the evaluation-at-α map $K\langle t_1, \dots, t_n\rangle \to \bar{K}$ is a bijection. The affinoid Nullstellensatz establishes this correspondence.

Having described the points of the polydisk, we can try giving it a topology. A natural first choice is to give $\mathbf{B}_K^n(\bar{K})$ the subspace topology induced by the product of the valuation topologies on $\bar{K}$. A basis is given by small polydisks

$$\{x \in \mathbf{B}_K^n(\bar{K}) : |x_i - a_i| < \epsilon_i \text{ for } 1 \leqslant i \leqslant n\},$$

where $a_i \in \bar{K}$ and $\epsilon_i > 0$. We then endow $\mathbf{B}_K^n$ with the quotient topology, called the *canonical topology*. The distinction between these two spaces matters: inequalities involving only functions with coefficients in K cannot separate two Galois-conjugate points upstairs. They do, however, give useful neighborhoods after we pass to the quotient; rational domains will give a basis downstairs.

Unfortunately, not all coverings in the canonical topology are ones over which we can glue analytic functions. We can already see the problem in dimension one. Consider the following decomposition of the disk $\mathbf{B}_K$ (taking $\operatorname{Aut}(\bar{K}/K)$-orbits): the following decomposition of $\mathbf{B}_K(\bar{K})$:

$$\mathbf{B}_K(\bar{K}) = \underbrace{\bigcup_{n \geqslant 1} \left\{x \in \bar{K} : |x| \leqslant |\pi|^{1/n}\right\}}_{\mathbf{D}_K(\bar{K})} \bigsqcup \underbrace{\{x \in \bar{K} : |x| = 1\}}_{\mathbf{T}_K(\bar{K})}$$

where $\pi \in K$ satisfies $0 < |\pi| < 1$, $\mathbf{D}$ is the *open disk* and $\mathbf{T}$ is the unit circle. This shows that the canonical topology on $\mathbf{B}_K$ is neither connected nor quasicompact, which is contrary to what we'd expect. Moreover, the sheaf theory doesn't work for this cover: We can write $\mathbf{B}_K(\bar{K}) = \mathbf{T}_K(\bar{K}) \sqcup \mathbf{D}_K(\bar{K})$, but it turns out that $\mathcal{O}(\mathbf{B}_K) \neq \mathcal{O}(\mathbf{D}_K) \times \mathcal{O}(\mathbf{T}_K)$. Here, $\mathcal{O}(-)$ is something like "convergent power series on $-$"; we'll make this precise later.

We will come back to the question of topology in a moment. For now, let us finish constructing our affinoid sets by considering closed subvarieties of $\mathbf{B}^n_K$. As in algebraic geometry, we can do this by taking quotients of the ring of functions.

DEFINITION 3.1. Let A be a K-algebra. A is an *affinoid* K-algebra if it is a quotient of $K\langle t_1, \dots, t_n\rangle$ for some n.

Many of the familiar finiteness properties of finitely generated K-algebras continue to hold for affinoid algebras. The next lemma makes this analogy precise.

There is one analytic input behind the following lemma. Tate's finiteness theorem says that $K\langle t_1, \dots, t_n\rangle$ is noetherian and Jacobson, that its maximal residue fields are finite over K, and that all of its ideals are closed for the Gauss norm

$$\left\| \sum_I a_I t^I \right\| = \max_I |a_I|.$$

These are consequences of Weierstrass division and the affinoid Nullstellensatz; we use them as standard input, rather than attempt to prove division here. See [**Con08**, Theorem 1.1.5], and the original treatment in [**Tat71**].

LEMMA 3.2. *Let A be an affinoid K-algebra. Then*

- *Any choice of surjection $K\langle t_1, \dots, t_n\rangle \to A$ defines a topology. Moreover, this topology is independent of presentation.*
- *A is noetherian.*
- *A is Jacobson.*
- *A prime ideal $\mathfrak{p}$ of A is maximal if and only if $A/\mathfrak{p}$ is finite dimensional over K.*
- *Every ideal of A is closed.*
- *Every homomorphism of K-algebras between affinoid algebras is continuous.*

PROOF. Choose a presentation $A = T_n/J$, where $T_n = K\langle t_1, \dots, t_n\rangle$. Since J is closed, the quotient norm makes A a Banach K-algebra. Noetherianity and the Jacobson property pass to quotients. A maximal ideal of A lifts to a maximal ideal of T_n, so its residue field is finite over K. Conversely, a finite dimensional K-algebra which is a domain is a field: multiplication by a nonzero element is an injective endomorphism of a finite dimensional vector space, hence is surjective. This proves the assertion about prime ideals. An ideal of A is closed because its inverse image in T_n is closed and A has the quotient topology.

Let us check continuity without assuming that the algebras are reduced. Give affinoid algebras A and B quotient Banach topologies, and let $\phi : A \to B$ be a K-algebra homomorphism. For a maximal ideal $\mathfrak{m} \subset B$ and $r \geqslant 1$, the algebra $B/\mathfrak{m}^r$ is finite dimensional over K: its filtration by powers of $\mathfrak{m}$ has finitely generated successive quotients over $B/\mathfrak{m}$. The composite

$$A \longrightarrow B/\mathfrak{m}^r$$

has a closed kernel, by the preceding paragraph. Its image is finite dimensional, so the induced map from the Banach quotient of A is continuous; here we use the uniqueness of the Hausdorff norm topology on a finite dimensional vector space over a complete valued field.

Suppose now that $a_j \to 0$ in A and $\phi(a_j) \to b$ in B. Applying the preceding maps gives $b \in \mathfrak{m}^r$ for every $\mathfrak{m}$ and r. The Krull intersection theorem in the noetherian local

ring $B_{\mathfrak{m}}$ gives $b/1 = 0$ in every $B_{\mathfrak{m}}$, and hence $b = 0$. Thus the graph of ϕ is closed. The Banach closed graph theorem, whose usual Baire-category proof works over our nontrivially valued complete field, implies that ϕ is continuous. Finally apply this to the identity map between the quotient topologies obtained from two presentations of A, in both directions. The topologies are the same, as required. □

For any ideal $I \subset K\langle t_1, \ldots, t_n\rangle$, we set

$$V(I) := \mathrm{MSpec}(K\langle t_1, \ldots, t_n\rangle/I) \cong \{\mathfrak{m} \in \mathbf{B}^n_K : \mathfrak{m} \supset I\} \subset \mathbf{B}^n_K$$

We can also define

$$V(I)(\bar{K}) := \{(x_1, \ldots, x_n) \in \mathbf{B}^n_K(\bar{K}) : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}$$

and then define $V(I)$ as the $\mathrm{Aut}(\bar{K}/K)$-quotient of $V(I)(\bar{K})$.

This construction endows $V(I)$ with the canonical topology. Given a ring map $\phi\colon A \to B$ of affinoid algebras, there is an induced map $\mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$ which is continuous for the canonical topology. Sets of the form $\mathrm{MSpec}(A)$ are called *affinoid sets*.

For any $x = \mathfrak{m} \in \mathrm{MSpec}(A)$, the quotient $A/\mathfrak{m}$ is a finite extension of K, and hence carries a unique valuation extending the valuation on K. Let $|f(x)|$ denote the real number corresponding to evaluating this absolute value on the element $f \bmod \mathfrak{m}$. Then, for a set of elements $\{f_1, \ldots, f_n, g\} \subset A$, one can make sense of the set

$$\{x \in \mathrm{MSpec}(A) : |f_i(x)| \leqslant |g(x)|\}$$

1.2. Fixing the Topology. Let us return to the issue we encountered with the disk. We would like to keep enough open sets to do geometry, but not allow coverings for which analytic gluing fails. In his original construction of rigid spaces, Tate was able to accomplish precisely this, by mimicking the construction of the structure sheaf $\mathcal{O}_{\mathrm{Spec}\, A}$ on $\mathrm{Spec}\, A$. Recall that one defines $\mathcal{O}_{\mathrm{Spec}\, A}$ in two steps: First, on distinguished opens $D(f)$, we set $\mathcal{O}_{\mathrm{Spec}\, A}(D(f)) = A_f$, and second, we show that this extends uniquely to a sheaf on all of $\mathrm{Spec}\, A$. Perhaps the most pivotal step in the above construction is using the quasicompactness of $\mathrm{Spec}\, A$ to reduce the computation to a *finite* distinguished cover.

This suggests that we look for a class of basic opens playing the role of $D(f)$, but flexible enough to retain both nonvanishing conditions and valuation inequalities (such as $\{f \neq 0\}$ and $\{|f| \leqslant 1\}$).

Definition 3.3. Let A be an affinoid K-algebra, and let $f_1, f_2, \ldots, f_m, g \in A$ generate the unit ideal. The *rational domain* associated to $f_1, \ldots, f_m, g$ is

$$U\left(\frac{f_1, \ldots, f_m}{g}\right) := \{x \in \mathrm{MSpec}(A) : |f_i(x)| \leqslant |g(x)|\}$$

Such sets form a basis for the canonical topology on $\mathrm{MSpec}(A)$, and these are the correct analogue of the $D(f)$'s in the rigid setting. Notice that $g(x) \neq 0$ on this domain: if $g(x) = 0$, all of the $f_i(x)$ would also vanish, contradicting the unit ideal condition.

Our next task is to specify the functions on a rational domain. The useful idea is to adjoin the fractions f_i/g as bounded analytic coordinates. We define

$$A\left\langle\frac{f_1, \ldots, f_m}{g}\right\rangle := A\langle s_1, \ldots, s_m\rangle/(gs_1 - f_1, \ldots, gs_m - f_m)$$

and put $\mathcal{O}_{\mathrm{MSpec}(A)}(U) = A\langle f_1/g, \dots, f_m/g\rangle$. This is again an affinoid algebra. To see why the fractions make sense, write $ag + \sum_i b_i f_i = 1$ in A. In the displayed quotient we have

$$g\left(a + \sum_i b_i s_i\right) = 1,$$

so g is invertible and $s_i = f_i/g$.

This description also explains why ordinary power series in the original coordinates are not enough. For example, on the annulus $|\pi| \leqslant |t| \leqslant 1$, the function t^{-1} is analytic, and its affinoid algebra is

$$K\langle t, s\rangle/(ts - \pi),$$

where $s = \pi/t$ and $t^{-1} = s/\pi$. A power series about $t = 0$ would miss this function.

More intrinsically, the rational localization is universal among maps to affinoid K-algebras for which g becomes invertible and every f_i/g becomes powerbounded. The latter means that its powers form a bounded set. Indeed, a convergent series in the s_i can then be evaluated, and the relations force a unique map. The affinoid maximum modulus theorem identifies powerboundedness with the inequalities $|f_i/g| \leqslant 1$ at all rigid points; this remains true in the presence of nilpotents [**Con08**, Theorem 1.2.6 and the discussion following it]. Consequently this universal property depends only on U, which proves independence of its chosen rational presentation.

Theorem 3.4 (Tate acyclicity, [**Tat71**]). *Let A be an affinoid algebra. Then, for any rational domain $U \subset \mathrm{MSpec}(A)$ and any finite covering $\{U_1, \dots, U_\ell\}$ of U by rational domains, the sequence*

$$0 \to \mathcal{O}_{\mathrm{MSpec}(A)}(U) \to \prod_i \mathcal{O}_{\mathrm{MSpec}(A)}(U_i) \to \prod_{i,j} \mathcal{O}_{\mathrm{MSpec}(A)}(U_i \cap U_j)$$

is exact.

In other words, the presheaf $\mathcal{O}_{\mathrm{MSpec}(A)}$ is a sheaf with respect to rational domains and finite coverings by rational domains. This is a fundamental analytic theorem, which we use as input. The last arrow sends $(s_i)_i$ to $(s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

So far we only know how to glue over finite covers by rational domains. We would also like to work with open subsets that are themselves affinoid, but need not be rational domains. These are the *affinoid domains*, which are roughly the rigid analytic analogue of affine open subschemes.

Definition 3.5. Let A be an affinoid K-algebra. $U \subset \mathrm{MSpec}(A)$ is an *affinoid domain* if there is an affinoid algebra B and a map of K-algebras $j\colon A \to B$ such that the induced map $j_*\colon \mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$ has image U and for which j is universal for such a property: If $h\colon A \to C$ is another K-algebra map of affinoid algebras, such that $h_*\colon \mathrm{MSpec}(C) \to \mathrm{MSpec}(A)$ has image contained in U, then h factors uniquely through j. Such a K-algebra B is unique up to unique isomorphism, if it exists.

Rational subsets are affinoid domains by the universal property just discussed. The assertion that every affinoid domain is a finite union of rational subsets is the Gerritzen–Grauert theorem, a further input from rigid geometry [**Con08**, Theorem 2.2.5]. Since B is essentially unique, to any affinoid domain $U \subset \mathrm{MSpec}(A)$, we may define $\mathcal{O}_{\mathrm{MSpec}(A)}(U) := B$.

THEOREM 3.6. *Let A be an affinoid algebra. Then, for any affinoid domain $U \subset \mathrm{MSpec}(A)$ and any finite covering $\{U_1, \dots, U_\ell\}$ of U by affinoid domains, the sequence*

$$0 \to \mathcal{O}_{\mathrm{MSpec}(A)}(U) \to \prod_i \mathcal{O}_{\mathrm{MSpec}(A)}(U_i) \to \prod_{i,j} \mathcal{O}_{\mathrm{MSpec}(A)}(U_i \cap U_j)$$

is exact.

PROOF. Write $B = \mathcal{O}(U)$, so that $U = \mathrm{MSpec}(B)$. The universal property makes each U_i an affinoid domain in $\mathrm{MSpec}(B)$. By the Gerritzen–Grauert theorem, choose a finite cover of each U_i by rational domains of $\mathrm{MSpec}(B)$. Such a rational domain is also rational after pullback to $\mathrm{MSpec}(\mathcal{O}(U_i))$, and its two affinoid algebras agree by their universal properties. Matching sections on the U_i therefore give matching sections on a finite rational cover of $\mathrm{MSpec}(B)$. Tate acyclicity glues them uniquely to an element of B. Applying uniqueness on the finite rational cover of each U_i shows that this element restricts to the original section there. The same argument proves injectivity. □

A general rigid space need not admit a finite affinoid cover. We should therefore allow opens that are unions of arbitrarily many affinoid domains. The trick is to require that the cover looks finite whenever we pull it back to an affinoid.

DEFINITION 3.7. An open subset $U \subset \mathrm{MSpec}(A)$ is *admissible* if there exists an affinoid cover $\{U_i\}$ of U such that for any map of affinoid K-algebras $\phi\colon A \to B$ such that $\phi_*(\mathrm{MSpec}(B)) \subset U$, the affinoid cover $\{\phi_*^{-1}(U_i)\}$ of $\mathrm{MSpec}(B)$ has a refinement by a finite open affinoid cover.

An *admissible cover* of an admissible open U is a collection of admissible open subsets $\{V_j\}$ of $\mathrm{MSpec}(A)$ which cover U and have the property that for any K-algebra map of K-affinoids $\psi\colon A \to B$ such that $\psi_*(\mathrm{MSpec}(B)) \subset U$, the cover $\{\psi_*^{-1}(V_j)\}$ has a refinement by a finite open affinoid cover.

Notice that this condition is precisely the sort of restriction we needed in our example of the disk. For an admissible open U, we define $\mathcal{O}(U)$ as the inverse limit of $\mathcal{O}(V)$ over the affinoid domains $V \subset U$, using the restriction maps we have already constructed.

THEOREM 3.8. *Let A be an affinoid algebra. Then, for any admissible open $U \subset \mathrm{MSpec}(A)$ and any admissible covering $\{U_i\}_{i \in I}$ of U by admissible opens, the sequence*

$$0 \to \mathcal{O}_{\mathrm{MSpec}(A)}(U) \to \prod_i \mathcal{O}_{\mathrm{MSpec}(A)}(U_i) \to \prod_{i,j} \mathcal{O}_{\mathrm{MSpec}(A)}(U_i \cap U_j)$$

is exact.

PROOF. Let (s_i) be a matching family and let $V \subset U$ be an affinoid domain. Admissibility gives a finite affinoid cover of V refining $\{V \cap U_i\}_i$. Restrict the s_i to this finite cover and apply the preceding theorem to obtain a unique section $s_V \in \mathcal{O}(V)$. A common finite affinoid refinement shows that s_V is independent of the chosen cover. The same uniqueness shows that the s_V are compatible as V varies. They therefore define an element of $\mathcal{O}(U)$, which restricts to the s_i. Uniqueness is checked on each V by the same finite-cover argument. This proves exactness. □

1.3. Gluing. We have now identified the open sets and coverings for which our sheaf theory works. Rather than trying to make these coverings come from an ordinary topology, we can simply specify which ones are allowed. This gives us a *G-topology*.

DEFINITION 3.9. A *G-topology*[1] on a topological space X is

- A collection $\mathrm{Perm}(X)$ of open subsets of X; these are called the *permissible opens*.
- For each $U \in \mathrm{Perm}(X)$, a set $\mathrm{Cov}(U)$ of open covers of U by permissible opens; these are the *permissible covers*.

These data satisfy the usual covering axioms: the identity cover is permissible, covers remain permissible after intersection with a permissible open, and a permissible cover followed by permissible covers of its members is again permissible. The permissible opens are stable under finite intersections.

Given a G-topology on X, we make the collection $\mathrm{Perm}(X)$ a category by declaring morphisms to be inclusions of open subsets.

DEFINITION 3.10. An *abelian presheaf* on the G-space X is a functor $\mathcal{F}\colon \mathrm{Perm}(X)^{\mathrm{op}} \to \mathrm{Ab}$. Here, Ab is the category of abelian groups. A presheaf $\mathcal{F}$ is a *sheaf* if for all $U \in \mathrm{Perm}(X)$ and $\{U_i\} \in \mathrm{Cov}(U)$, the following sequence is exact:

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

DEFINITION 3.11. Given a G-space X and an abelian sheaf $\mathcal{F}$ on X, the *stalk* of $\mathcal{F}$ at a point $x \in X$ is the direct limit

$$\mathcal{F}_x := \varinjlim_{U \in N(x)} \mathcal{F}(U)$$

where $N(x)$ is the set of permissible open subsets containing x.

DEFINITION 3.12. A *locally ringed G-space* is a pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of rings whose stalks are local rings. If $\mathcal{O}_X$ is a sheaf of K-algebras, we say X is a locally ringed G-space *over* K.

From the previous constructions, to any affinoid algebra A, we can associate two locally ringed G-spaces over K, given as the pair $\mathrm{Sp}(A) := (\mathrm{MSpec}(A), \mathcal{O}_{\mathrm{MSpec}(A)})$, where the permissible opens are the admissible ones (resp. affinoid domains) and the permissible covers are the admissible ones (resp. finite affinoid covers). These are called the strong and weak G-topologies on $\mathrm{MSpec}(A)$, respectively. Unless otherwise specified, we will consider the *strong* topology on $\mathrm{MSpec}(A)$. A locally ringed G-space over K of the form $(\mathrm{MSpec}(A), \mathcal{O}_{\mathrm{MSpec}(A)})$ is called an *affinoid space* over K.

One useful feature of affine schemes is that maps between them are determined by maps of rings. The same is true of our affinoid spaces.

LEMMA 3.13. *Let A and B be affinoid K-algebras. Global sections give a bijection*

$$\mathrm{Hom}(\mathrm{Sp}(B), \mathrm{Sp}(A)) \longrightarrow \mathrm{Hom}_K(A, B).$$

[1] There are more technical conditions than we list here; we refer the interested reader to [Tat71] for more details. Since we will not use G-topologies for the rest of this thesis, we feel that this rough definition should suffice.

Proof. Let $\phi : A \to B$ be a K-algebra map. For $\mathfrak{n} \in \mathrm{MSpec}(B)$, the ring $A/\phi^{-1}(\mathfrak{n})$ embeds into the finite extension $B/\mathfrak{n}$ of K. It is therefore a finite dimensional domain over K, hence a field. Thus

$$f : \mathrm{MSpec}(B) \longrightarrow \mathrm{MSpec}(A),\ \mathfrak{n} \longmapsto \phi^{-1}(\mathfrak{n})$$

is defined. The inverse image of a rational domain is the rational domain obtained by applying ϕ to its defining functions. The universal property of rational localization gives the corresponding maps on sections. These commute with restriction, and the sheaf property extends them to a morphism of locally ringed G-spaces. Pullbacks of admissible covers are admissible by their definition.

Conversely, a morphism of locally ringed G-spaces gives a map $\phi : A \to B$ on global sections. Locality forces its map on points to be the one above: the kernel of evaluation at $f(\mathfrak{n})$ is the inverse image of the kernel of evaluation at $\mathfrak{n}$. On every rational domain its map on functions is then forced by the rational-localization universal property. Rational domains form a basis for the G-topology, so the entire morphism is determined by ϕ. These two constructions are inverse. □

Given any locally ringed G-space $(X, \mathcal{O}_X)$ over K, and any permissible open subset $U \subset X$, there is a natural G-topology on U, given by taking all of the elements of $\mathrm{Perm}(X)$ contained in U. Moreover, we can restrict $\mathcal{O}_X|_U =: \mathcal{O}_U$ so that the pair $(U, \mathcal{O}_U)$ is a locally ringed G-space over K.

Now we can glue affinoid spaces along permissible opens, just as we glue affine schemes along open subschemes, to define general rigid spaces.

Definition 3.14. A rigid analytic variety over K is a pair $(X, \mathcal{O}_X)$ which is a locally ringed G-space over K such that there is a permissible cover $\{U_i\}$ of X such that $(U_i, \mathcal{O}_{U_i})$ is isomorphic to an affinoid space. A morphism of rigid spaces is a morphism of locally ringed G-spaces over K.

In fact, once we have Lemma 3.13, we can also describe maps from an arbitrary rigid space into an affinoid space: we simply construct them on affinoid charts and glue.

Lemma 3.15. *Let X and Y be rigid analytic varieties with Y affinoid. Then the natural map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}_K(\mathcal{O}(Y), \mathcal{O}(X))$ is a bijection.*

Proof. Write $Y = \mathrm{Sp}(A)$ and choose an admissible affinoid cover $X = \bigcup_i X_i$, with $X_i = \mathrm{Sp}(B_i)$. A homomorphism $A \to \mathcal{O}_X(X)$ restricts to maps $A \to B_i$, hence to morphisms $X_i \to Y$ by Lemma 3.13. On every affinoid domain in $X_i \cap X_j$, the two morphisms induce the same map on sections. The uniqueness in Lemma 3.13 makes them equal there, so they glue to a morphism $X \to Y$. Conversely, two morphisms inducing the same map on global sections agree on each X_i, again by that lemma. This proves both surjectivity and injectivity. □

2. Adic Spaces

For much of what we have done so far, rigid spaces are perfectly satisfactory. In the next chapter, however, we will encounter perfectoid spaces, whose rings of functions are highly non-noetherian and far from being of finite type. We need a setting in which we can work with both kinds of spaces. This is provided by Huber's theory of adic

spaces [Hub93, Hub94, Hub96]. Roughly speaking, adic spaces are to rigid spaces as schemes are to varieties.

There is a second advantage to Huber's construction. Instead of the set of rigid points, Huber considers valuations, including valuations of higher rank. The extra points make it possible to work with an actual topological space, rather than having to specify admissible covers separately.

2.1. Valuation Spectra. For the remainder of this thesis, all topological rings and modules will be assumed to be *linearly topologized*. This means that the topology is Hausdorff, and given by a fundamental system of open subgroups.

The valuations we have considered so far have taken values in the positive real numbers. There is no need to restrict ourselves to the real numbers, however: we can instead allow arbitrary totally ordered abelian groups. This is where the additional points in Huber's construction come from.

Definition 3.16. A valuation on a ring A is a map $|\cdot|: A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group (written multiplicatively) such that:

- $|xy| = |x||y|$.
- $|x + y| \leqslant \max\{|x|, |y|\}$.
- $|0| = 0$ and $|1| = 1$.

We always take Γ to be generated by the nonzero values of the valuation. Its *support* $\{a \in A : |a| = 0\}$ is a prime ideal, and the valuation extends uniquely to the fraction field of the quotient by its support.

Two valuations $|\cdot|$ and $|\cdot|'$ are said to be *equivalent* if, for all $a, b \in A$, we have $|a| \leqslant |b|$ if and only if $|a|' \leqslant |b|'$. We denote the set of all equivalence classes of valuations on A by $\mathrm{Spv}(A)$. This set is topologized with the topology generated by sets of the form $\{x \in \mathrm{Spv}(A) : |a|_x \leqslant |b|_x \neq 0\}$.

Now let A be a topological ring. A valuation $|\cdot|: A \to \Gamma \cup \{0\}$ is *continuous* if the subset

$$\{a \in A : |a| < \gamma\} \subset A$$

is open in A for all $\gamma \in \Gamma$. The set of all continuous valuations on A is denoted $\mathrm{Cont}(A) \subset \mathrm{Spv}(A)$, and is given the subspace topology.

Definition 3.17. Let R be a ring and $I \subset R$ be an ideal satisfying $\bigcap_{n \geqslant 0} I^n = 0$. The I-adic topology on R is the linear topology with $\{I^n : n \geqslant 0\}$ as a fundamental system of open neighborhoods of 0. We say a topological ring R is *adic* if its topology coincides with the I-adic topology for some ideal $I \subset R$.

Definition 3.18. A topological ring A is a *Huber ring* if there exists an open subring $A_0 \subset A$ which is adic with respect to a finitely generated ideal $I \subset A_0$. We say A_0 is a "ring of definition" and I is an ideal of definition.

Definition 3.19. Let A be a Huber ring with ideal of definition I. A subset $S \subset A$ is *bounded* if there is some (A_0, I) such that $I^n \cdot S \subset A_0$ for some $n \in \mathbf{N}$.

Proposition 3.20. *Let A be a Huber ring. A subring $A' \subset A$ is a ring of definition if and only if A' is open and bounded.*

Proof. A ring of definition is open by definition and bounded by taking itself as A_0. Conversely, suppose that A' is open and bounded. Choose a ring of definition A_0 and a finitely generated ideal of definition I such that

$$I^n A' \subset A_0$$

for some n. Openness gives an $m \geqslant 1$ with $I^m \subset A'$. Choose generators $f_1, \dots, f_r$ of I^m as an A_0-ideal, and put $J = (f_1, \dots, f_r)A'$. Then J is a finitely generated ideal of A'. For $m\ell \geqslant n$, we have

$$I^{m(\ell+1)} \subset J^\ell \subset I^{m\ell} A' \subset I^{m\ell - n} A_0.$$

For the first inclusion, write $I^{m(\ell+1)} = (I^m)^\ell I^m$ and absorb the final factor, including its A_0-coefficients, into A'. These inclusions show that the J-adic neighborhoods and the subspace neighborhoods of zero in A' are cofinal. Thus A' is adic for a finitely generated ideal, as required. □

Theorem 3.21 (Huber, [**Hub93**, Corollary 3.2]). *If A is a Huber ring, then* $\mathrm{Cont}(A)$ *is a spectral space. Moreover, for $a \in A$, the set $\{x : |a(x)| \leqslant 1\}$ is constructible in* $\mathrm{Cont}(A)$.

Recall that a spectral space is a topological space X that is quasicompact, with the property that it has a basis of quasicompact opens that are stable under finite intersection, and every irreducible closed subset $Z \subset X$ has a unique generic point. One of the main results in Hochster's thesis provides a nice classification of spectral spaces:

Theorem 3.22 (Hochster [**Hoc69**]). *Let X be a topological space. The following are equivalent:*

- $X \cong \mathrm{Spec}(A)$ *for some ring A.*
- $X \cong \varprojlim X_i$, *where each X_i is a finite T_0 space.*
- X *is a spectral space.*

The nice thing about spectral spaces is that we are already familiar with their topology from algebraic geometry. In particular, the basis of quasicompact opens lets us reduce gluing questions to finite covers, something we could not do in the canonical topology on the rigid points.

Although $\mathrm{Cont}(A)$ is spectral, it is not quite the space we want[2]. We still have to decide which elements of A we regard as integral. We will only keep those valuations for which the integral elements have value at most 1.

Definition 3.23. Let A be a Huber ring. A subring $A^+ \subset A$ is a *ring of integral elements* if it is open and integrally closed in A, and its elements are powerbounded. We say an element $a \in A$ is powerbounded if the set $\{a^n\}_{n \geqslant 1}$ is bounded in the sense of Definition 3.19.

We write A° for the subring of powerbounded elements of A.

A pair (A, A^+), where A is a Huber ring and $A^+ \subset A$ is a ring of integral elements, is called a *Huber pair*. A morphism of Huber pairs $(A, A^+) \to (B, B^+)$ is a continuous homomorphism $f : A \to B$ such that $f(A^+) \subset B^+$.

Definition 3.24. Let (A, A^+) be a Huber pair. We define the *adic spectrum* $\mathrm{Spa}(A, A^+)$ to be the subspace

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Cont}(A) : |a|_x \leqslant 1 \text{ for all } a \in A^+\} \subset \mathrm{Cont}(A)$$

[2] See the appendix to Lecture 3 in [**SW20**] for a detailed discussion.

LEMMA 3.25. $\mathrm{Spa}(A, A^+)$ *is a spectral space.*

PROOF. Let $Y = \mathrm{Cont}(A)$. The constructible, or patch, topology on a spectral space is the compact Hausdorff topology generated by its quasicompact opens and their complements. By Huber's theorem, each condition $|a(x)| \leqslant 1$ is closed in this topology. Hence

$$\mathrm{Spa}(A, A^+) = \bigcap_{a \in A^+} \{x \in Y : |a(x)| \leqslant 1\}$$

is patch closed, or *pro-constructible*.

Let us recall why this gives the conclusion. If Z is patch closed in a spectral space, intersections with quasicompact opens of the ambient space are compact in the patch topology, hence quasicompact in the original topology. They form a basis stable under finite intersections. To check sobriety, take an irreducible closed subset C of Z. Its closure in the ambient space has a generic point η. Every patch neighborhood of η meets C: if a basic one is $U \backslash V$, where U, V are quasicompact opens, then U meets C, while $\eta \notin V$ implies that V misses the closure of C. Thus η lies in the patch closure of C, which is contained in Z, and is its unique generic point. Applying this to the displayed subset proves that $\mathrm{Spa}(A, A^+)$ is spectral. □

Let us see what these spaces look like in a few familiar cases.

EXAMPLE 3.26. If A is any ring with the discrete topology, then A is a Huber ring, with ideal of definition (0). In this trivial case, we have $\mathrm{Cont}(A) = \mathrm{Spv}(A)$.

EXAMPLE 3.27. Let A be a complete, separated I-adic ring, where I is finitely generated. Here A itself is a ring of definition and a ring of integral elements, so (A, A) is a Huber pair. There is a natural injection of underlying topological spaces

$$|\mathrm{Spf}(A)| = |\mathrm{Spec}(A/I)| \longrightarrow \mathrm{Spa}(A, A),$$

sending a prime $\mathfrak{p} \supset I$ to the valuation which is 0 on $\mathfrak{p}$ and 1 elsewhere. This valuation is continuous because its support is open. Conversely, a prime ideal is open precisely when it contains I, since containing a power of I is equivalent to containing I for a prime. On these trivial valuations, a basic inequality $|a| \leqslant |b| \neq 0$ cuts out $D(b)$, so the map is a topological embedding. We are identifying the underlying space of the formal spectrum here, not its structure sheaf with that of the adic spectrum.

REMARK 3.28. Let A be a non-archimedean field K, so that its topology is defined by a nontrivial rank one valuation $|\cdot| : K \to \mathbf{R}_{>0} \bigcup \{0\}$. Then $\mathrm{Cont}(K)$ can have more than one point - that is, it can have a point x such that $|\cdot|_x \not\sim |\cdot|$. This is one advantage of Huber's theory - it allows us to see higher rank valuations!

DEFINITION 3.29. An *affinoid field* is a Huber pair (K, K^+), where K is a field equipped with the discrete topology or a nonarchimedean field, and $K^+ \subset K$ is an open and bounded valuation ring of K.

PROPOSITION 3.30. *Let (K, K^+) be an affinoid field as above. Then* $\mathrm{Spa}(K, K^+)$ *is the set of open and bounded valuation subrings $R \subset K$ such that $K^+ \subset R$. In particular, if K is a nonarchimedean field, then* $\mathrm{Spa}(K, \mathcal{O}_K)$ *is a one-point set.*

PROOF. Equivalence classes of valuations on a field are determined by their valuation rings. Explicitly, the valuation attached to a valuation ring $R \subset K$ has value group

$K^\times/R^\times$, ordered so that the elements of R have value at most 1. The condition defining $\mathrm{Spa}(K, K^+)$ says exactly that $K^+ \subset R$.

If K is discrete, every valuation is continuous and every subset is open and bounded, so there is nothing more to prove. Suppose that K is nonarchimedean, and let v be a continuous valuation on it. Its valuation ring R is open, since its additive subgroup contains the open neighborhood $\{a : v(a) < 1\}$ of zero. Moreover $R \subset \mathcal{O}_K$: if $a \in R$ had $|a|_K > 1$, then $a^{-n} \to 0$ in K, whereas $v(a^{-n}) \geqslant 1$ for every n, contradicting continuity. Thus R is bounded.

Conversely, let R be an open bounded valuation ring of K. Boundedness and closure under powers imply $R \subset \mathcal{O}_K$. Choose a sufficiently small nonzero $\pi \in R$ with $|\pi|_K < 1$. Since $\pi^{-1} \notin R$, we have $v_R(\pi) < 1$. The maximal ideal $\mathfrak{m}_R$ contains the open subgroup πR, so it is open. Every strict valuation ball is of the form $c\mathfrak{m}_R$ for some $c \in K^\times$, and is therefore open. Hence v_R is continuous. Finally, if $K^+ = \mathcal{O}_K$, the inclusions $\mathcal{O}_K \subset R \subset \mathcal{O}_K$ force $R = \mathcal{O}_K$. □

In particular, we recover a one-point space for a nonarchimedean field by choosing the right ring of integral elements, just as we would expect. It remains to equip the adic spectrum with its structure sheaf.

2.2. (Pre)sheaves. Recall how we defined the structure sheaf in rigid geometry: we first described the functions on rational domains and then used these to describe the functions on more general opens. We can do much the same thing on $\mathrm{Spa}(A, A^+)$, where the counterparts of the rational domains (and of the opens $D(f) \subset \mathrm{Spec}(A)$) are called *rational subsets*.

Definition 3.31. Let $X = \mathrm{Spa}(A, A^+)$. A rational subset $U \subset X$ is a subset of the form

$$U = U\left(\left\{\frac{T_j}{s_j}\right\}_{j\in J}\right) = \{x \in X : |t(x)| \leqslant |s_j(x)| \neq 0 \text{ for all } t \in T_j, j \in J\}$$

Here, J is some finite set, $s_j \in A$ and each $T_j \subset A$ is a finite subset of A such that $T_jA \subset A$ is open in A.

The rational subsets of X form a basis of quasicompact opens, stable under finite intersections [**Hub93**, Theorem 3.5].

Theorem 3.32 (Huber, [**Hub94**, Proposition 1.3]). *Let $U \subset X$ be a rational subset. There is a complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$, together with a map $f : (A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ such that:*

- $\mathrm{Spa}(f) : \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X(U)^+) \to X$ *is a homeomorphism onto U.*
- *If (C, C^+) is a complete Huber pair and $g : (A, A^+) \to (C, C^+)$ is a morphism such that the induced map on adic spectra $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(A, A^+)$ has image contained in U, then g factors uniquely as $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X(U)^+) \to (C, C^+)$ for some $h : (\mathcal{O}_X(U), \mathcal{O}_X(U)^+) \to (C, C^+)$.*

Construction and universal property. It suffices to treat $U = U(T/s)$; finite intersections are handled by repeating the construction. Choose a ring of definition A_0 with ideal of definition I. On $A[1/s]$, use the topology with ring of definition

$$A_0[T/s] = A_0[t/s : t \in T]$$

and ideal of definition $IA_0[T/s]$. The openness of TA ensures that this is a ring topology: after multiplying the finitely many coefficients expressing a power of I in TA by a further power of I, one obtains $I^N \subset TA_0$ for some N, and hence $s^{-1}I^N \subset A_0[T/s]$. The intermediate topology need not be separated; let $A\langle T/s\rangle$ be its separated completion. Its ring of integral elements is the closure of the integral closure of $A^+[T/s]$ in $A[1/s]$.

We use two valuation-theoretic facts of Huber: a complete Huber pair is zero if and only if its spectrum is empty, and its integral subring consists precisely of the elements whose values at all points are at most 1 [**Hub93**, Proposition 3.6 and Lemma 3.3]. They imply that in a map to (C, C^+) as in the statement, s is a unit and $t/s \in C^+$ for every $t \in T$.

To see why s must be a unit, suppose the contrary. Then the separated completion of $C/(s)$ would be nonzero and would supply a point where s vanishes. Indeed, if 1 were in the closure of (s), some multiple of s would be sufficiently close to 1 to be a unit. The resulting continuous map $A[1/s] \to C$ extends uniquely to its completion and preserves the integral subrings. This proves the claimed factorization.

The remaining valuation-theoretic input is that passing to completion does not change an adic spectrum [**Hub93**, Proposition 3.9]. Before completion, extending a valuation to $A[1/s]$ with the chosen topology and integral subring amounts exactly to requiring $|t| \leqslant |s| \neq 0$ for $t \in T$. This identifies the spectrum with U, including its rational-open topology, and proves the theorem. □

We know, by the universal property, how to restrict functions from one rational subset to another. So a section over a general open should just be a compatible family of sections over the rational subsets it contains.

Definition 3.33. We define a presheaf of complete topological rings on X by

$$\mathcal{O}_X(W) = \varprojlim_{U \subset W} \mathcal{O}_X(U)$$

for any open subset W of X. The limit runs over all rational subsets contained in W. We make the analogous definition for $\mathcal{O}_X^+$.

Of course, defining a presheaf is not quite the same as showing that it is a sheaf. But we can already say something about its stalks.

Proposition 3.34. *Fix the notation above. Then:*

- *For all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring, and the valuation $|\cdot|_x$ extends to a valuation on $\mathcal{O}_{X,x}$.*
- $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} : |f(x)| \leqslant 1\}$.

Proof. For a rational neighborhood U of x, the rational-localization theorem identifies x with a valuation on $\mathcal{O}_X(U)$. These valuations are compatible with restriction. They therefore define a valuation on the direct limit $\mathcal{O}_{X,x}$.

Let f be a germ represented in $B = \mathcal{O}_X(U)$ and suppose that $|f(x)| \neq 0$. Choose a ring of definition B_0 and a finitely generated ideal of definition J. By continuity of the valuation, some power J^N has all its values strictly smaller than $|f(x)|$. If $h_1, \dots, h_r$ generate J^N, the rational neighborhood

$$W = \{y \in U : |h_i(y)| \leqslant |f(y)| \neq 0 \text{ for every } i\}$$

contains x, and its localization inverts f. Thus the germ of f is a unit. Conversely, a germ with value zero cannot be a unit. The nonunits are therefore exactly the kernel of the valuation, an ideal, proving that the stalk is local.

A germ from $\mathcal{O}^+_{X,x}$ has value at most 1 by construction. Conversely, if a germ represented by $f \in \mathcal{O}_X(U)$ satisfies $|f(x)| \leqslant 1$, restrict to the rational neighborhood $\{y \in U : |f(y)| \leqslant 1\}$. Its rational-localization construction includes f in the ring of integral elements. This proves the second assertion. □

Definition 3.35. We say a Huber pair (A, A^+) is *sheafy* if the structure presheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ is a sheaf of complete topological rings. If (A, A^+) is sheafy, we say $\mathrm{Spa}(A, A^+)$ is an *affinoid adic space*.

Whether the structure presheaf is actually a sheaf might appear to depend on both parts of the Huber pair. In the Tate case that we will use, however, it depends only on the ring A, not on the choice of A^+.

Proposition 3.36 (Independence of integral elements in the Tate case). *Suppose that A contains a unit whose powers converge to zero. Then the sheafiness of (A, A^+) is independent of A^+.*

Proof. We may replace A by its completion. Since a power of a topologically nilpotent unit belongs to every open ideal, every open ideal of A is the unit ideal. Thus rational localizations are specified by finite sets of elements generating the unit ideal, and their underlying complete rings depend only on A.

The covering input we need is Huber's refinement lemma: every rational cover of a rational subset admits a refinement by a *standard rational cover*. For a rational localization B and elements $f_1, \dots, f_r$ generating B, this is the cover

$$\left\{ U\left(\frac{f_1, \dots, f_r}{f_i}\right) \right\}_{1 \leqslant i \leqslant r}.$$

See [**Hub94**, Lemma 2.6] and [**Ked17**, Lemma 1.6.8]. Such sets cover because at each valuation one of the f_i has maximal, nonzero value. The collection of these covers, and the rings and restriction maps in their equalizer diagrams, depend only on A, not on its integral subring.

The sheaf axiom can be tested on a cofinal family of finite rational covers. To see this directly, refine a cover to one in that family. Uniqueness on the refinement first proves separatedness; a matching family then glues on the refinement, and separatedness shows that the glued section agrees with the original sections. Applying this on every rational subset and taking the defining inverse limits proves the axiom for arbitrary open covers. Thus sheafiness is equivalent to exactness of the same ring-theoretic diagrams for every choice of A^+, which proves the assertion. This is the covering argument recorded in [**Ked17**, Remark 1.6.9]. □

This is the case we will need for analytic geometry. The hypothesis is visible because the standard-cover refinement argument uses the fact that open ideals are the unit ideal.

2.3. Gluing. We are almost ready to define general adic spaces. The only thing to keep in mind when gluing our affinoids is that, besides the topology and the structure sheaf, we also want to remember the valuations at the points.

Definition 3.37. The category $\mathcal{V}$ of v-ringed spaces has as objects triples $X = (|X|, \mathcal{O}_X, \{|\cdot|_x\}_{x\in|X|})$, where:

- $|X|$ is a topological space.
- $\mathcal{O}_X$ is a sheaf of complete topological rings on $|X|$.
- $|\cdot|_x$ is a continuous valuation on $\mathcal{O}_{X,x}$, with the direct-limit topology on the stalk.

Morphisms are pairs (f, ϕ), where:

- $f : |X| \to |Y|$ is a continuous map.
- $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of complete topological rings such that for all $x \in X$, the valuation $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \to \Gamma_{|\cdot|} \cup \{0\}$ is equivalent to the valuation $|\cdot|_{f(x)}$.

The valuations determine a subsheaf

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leqslant 1 \text{ for all } x \in U\}.$$

This category is well suited to gluing affinoid adic spaces. For affinoid adic spaces the stalks are local and the supports of the valuations are their maximal ideals, by the preceding proposition. Hence morphisms between adic spaces in this category are automatically local on stalks.

Definition 3.38. An adic space is an object $X \in \mathcal{V}$ which has an open covering by affinoid adic spaces.

Proposition 3.39. *If (A, A^+) is sheafy and X is an adic space, then*

$$\mathrm{Hom}(X, \mathrm{Spa}(A, A^+)) \cong \mathrm{Hom}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X)))$$

Proof. On the right, Hom means continuous ring homomorphisms $A \to \mathcal{O}_X(X)$ carrying A^+ into $\mathcal{O}_X^+(X)$; the global sections need not themselves be a Huber ring. A morphism of adic spaces gives such a homomorphism by taking global sections and using the completion map $A \to \widehat{A}$.

Conversely, let ϕ be such a homomorphism. For $x \in X$, composition with the valuation at x gives

$$A \xrightarrow{\phi} \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x} \xrightarrow{|\cdot|_x} \Gamma_x \cup \{0\}.$$

This is a continuous valuation bounded by 1 on A^+, so it defines a point $f(x) \in \mathrm{Spa}(A, A^+)$. On any affinoid chart of X, inverse images of basic valuation inequalities are open. Thus f is continuous.

Let U be a rational subset of the target. Cover $f^{-1}(U)$ by rational subsets V of affinoid charts of X. The map $A \to \mathcal{O}_X(V)$ preserves integral elements and its spectrum maps into U, so the rational-localization universal property gives a unique continuous map $\mathcal{O}(U) \to \mathcal{O}_X(V)$. These maps agree after restriction, again by uniqueness. They therefore glue to a map $\mathcal{O}(U) \to \mathcal{O}_X(f^{-1}(U))$. Repeating this on the rational basis gives a morphism of sheaves, and the definition of $f(x)$ gives compatibility with the valuations. Finally, both the map on points and the maps on a rational basis are forced by ϕ, proving uniqueness and the desired bijection. □

Proposition 3.40. *If (A, A^+) is sheafy and $X = \mathrm{Spa}(A, A^+)$, then $\mathcal{O}_X(X) = \widehat{A}$.*

Proof. The entire space is the rational subset $U(1/1)$. Its rational localization is the completion of A, since adjoining $1/1$ changes neither the ring nor its topology. In the inverse-limit definition of $\mathcal{O}_X(X)$, a compatible family is determined by its component on this largest rational subset, and every element of $\widehat{A}$ gives such a family by restriction. In fact this calculation already holds for the structure presheaf; sheafiness lets us regard it as the ring of global sections of an adic space. □

2.4. Sheafy Rings. It remains to give some examples of rings for which the structure presheaf really is a sheaf. We begin with Tate rings, which will appear throughout the rest of the thesis.

Definition 3.41. A Huber ring A is *Tate* if it contains a topologically nilpotent unit ϖ, that is, a unit ϖ such that $\lim_{n\to\infty} \varpi^n = 0$.

Definition 3.42. If A is a Tate ring, we define

$$A\langle T_1, \dots, T_n\rangle := \left\{ f = \sum_{I\in\mathbf{N}^n} a_I T^I \in \hat{A}[[T_1, \dots, T_n]] : a_I \to 0 \text{ as } |I| = \sum_{k=1}^{n} i_k \to \infty \right\}$$

Definition 3.43. Let A be a Tate ring.

- We say A is *strongly noetherian* if $A\langle T_1, \dots, T_n\rangle$ is noetherian for all $n \geqslant 0$.
- We say A is *uniform* if the subring of power-bounded elements $A^\circ \subset A$ is bounded.
- We say A is *stably uniform* if $\mathcal{O}_X(U)$ is uniform for all rational subsets $U \subset X = \mathrm{Spa}(A, A^\circ)$.
- We say A is *perfectoid* if it is a complete, uniform $\mathbf{Z}_p$-algebra for some fixed p, such that there is a topologically nilpotent unit ϖ with $pA^\circ \subset \varpi^p A^\circ$, and the Frobenius $\Phi\colon A^\circ/\varpi \to A^\circ/\varpi^p$ is surjective.

Proposition 3.44 (Huber, Buzzard-Verberkmoes, Scholze). *Let (A, A^+) be a complete Huber pair. It is sheafy in each of the following cases:*

- *A is discrete.*
- *A is finitely generated as an algebra over a noetherian ring of definition.*
- *A is a strongly noetherian Tate ring.*
- *A is a stably uniform Tate ring.*
- *A is perfectoid.*

Proof. First, let us isolate what must be proved. Rational subsets form a quasicompact basis stable under intersection. Thus it is enough that, for every finite rational cover $U = \bigcup_i U_i$, the diagram

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \prod_i \mathcal{O}(U_i) \longrightarrow \prod_{i,j} \mathcal{O}(U_i \cap U_j)$$

be an equalizer, with the indicated topology. Once this holds on the basis, the inverse-limit construction gives the sheaf axiom on arbitrary opens: restrict a matching family to a finite rational refinement of each rational subset, glue there, and use uniqueness to obtain compatible sections in the inverse limit.

For a discrete ring, a rational localization with denominator s is just $A[1/s]$ with the discrete topology. If rational subsets with denominators s_i cover one with denominator s, then the principal opens $D(s_i)$ cover $D(s)$. Indeed, for every prime $\mathfrak{p}$ not containing

s, the trivial valuation with support $\mathfrak{p}$ belongs to the original rational subset, so belongs to one of the covering subsets. The required equalizer is therefore the usual localization equalizer for the affine scheme $\mathrm{Spec}(A[1/s])$. Intersections correspond to inverting the products of the denominators, so this proves the discrete case.

For the next two cases we use Huber's acyclicity theorem and its proof [**Hub94**, Theorems 2.2 and 2.5]: when a complete Huber ring is finitely generated over a noetherian ring of definition, or is Tate and strongly noetherian, the augmented Čech complex for every finite rational cover is exact, and its initial map is a topological embedding. Its first two exactness assertions are precisely the equalizer condition above, including its topology. The noetherian hypotheses enter this analytic input through control of ideals and completion; an arbitrary completion does not preserve exact sequences.

For a stably uniform Tate ring, we use the local calculation behind [**BV18**, Theorem 7], in the generality of Tate rings as proved in [**Ked17**, Lemma 1.7.3]: a uniform complete Tate ring satisfies the equalizer condition for a simple Laurent cover $\{|f| \leqslant 1\} \cup \{|f| \geqslant 1\}$. The estimate behind that calculation is that boundedness of the powerbounded subring controls the kernels of the maps from restricted power series to the two rational localizations.

To pass from this special cover to general covers, use the fact that every finite rational cover admits a refinement by successive Laurent covers [**Ked17**, Remark 1.6.14]. Stable uniformity makes the same calculation available after each rational localization. Gluing successively along the refinement proves the equalizer condition; uniqueness then descends it to the original cover, by the argument used in the preceding proposition.

Finally, Scholze's rational-localization theorem says that every rational localization of a perfectoid ring is again perfectoid, in particular uniform; in the generality of the definition above, see [**SW20**, Theorem 7.1.1]. Thus a perfectoid ring is stably uniform and the previous case applies. This last input is part of the perfectoid theory developed in the next chapter; we are recording its consequence for the present construction, not using the present proposition to prove it. □

The strongly noetherian case includes the familiar affinoid K-algebras that we encountered in our discussion of rigid geometry.

Example 3.45. Let K be a nonarchimedean field. Then $K\langle T_1, \ldots, T_n\rangle$ is strongly noetherian, and in fact, any quotient of $K\langle T_1, \ldots, T_n\rangle$ is strongly noetherian. Such quotients are called K-algebras of *topologically finite type* or *tft* for brevity.

2.5. Rigid Spaces. We have now encountered the rings from which rigid spaces are built within Huber's theory. Let us make the relationship between the two definitions precise. For the rest of this subsection, fix a nonarchimedean field K.

Definition 3.46. An adic space over $\mathrm{Spa}(K, K^\circ)$ is *locally tft* over K if it admits an open cover by affinoid adic spaces $\mathrm{Spa}(A_i, A_i^\circ)$ such that each A_i is a tft K-algebra and the identifications respect the maps to $\mathrm{Spa}(K, K^\circ)$.

The category of locally tft adic spaces over K is a full subcategory of the category of adic spaces over $\mathrm{Spa}(K, K^\circ)$. The precise comparison with the usual category of rigid spaces is an equivalence on quasi-separated objects. Here quasi-separated means that the intersection of two affinoid opens has a finite affinoid cover; on the rigid side this

cover is required to be admissible. This condition makes the overlaps manageable when we glue.

THEOREM 3.47 (Huber, [Hub94, Proposition 4.5]). *There is an equivalence of categories*

$$\left\{\begin{array}{c}\text{quasi-separated rigid analytic}\\ \text{spaces over } K\end{array}\right\} \cong \left\{\begin{array}{c}\text{quasi-separated adic spaces}\\ \text{locally tft over } K\end{array}\right\}.$$

Without quasi-separatedness, the rigid-to-adic functor is still fully faithful.

EXPLANATION OF THE COMPARISON. The functor on affinoids is $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^\circ)$. The basic comparison input is that rational domains on the two sides correspond, with the same affinoid algebras, and that a finite rational family covers on one side precisely when it covers on the other [Hub93, Corollaries 4.2–4.4]. Full faithfulness on affinoids follows from Lemma 3.13 and the analogous global-sections property for adic spectra proved above. Maps of affinoid K-algebras preserve powerbounded elements because they are continuous.

An admissible affinoid cover and its transition maps therefore give an open cover by adic affinoids with the same gluing data. Conversely, a quasi-separated adic space locally of the stated form gives affinoid rigid charts. Quasi-separatedness gives finite affinoid covers of their overlaps, so the rational-basis comparison makes the overlap data admissible on the rigid side. Gluing gives inverse constructions on the quasi-separated subcategories. The same local full faithfulness makes them inverse on morphisms as well. □

We can therefore work with rigid spaces using Huber's language without losing any of the relevant geometry. The advantage for us is that we now have actual topological spaces and can, for instance, study sheaves by looking at their stalks.

THEOREM 3.48 (Huber, [Hub93, Corollary 4.4]). *Given a tft K-algebra A, the topos of sheaves of sets on its Tate site is the topos of sheaves on the sober space*

$$\mathcal{T}_A = |\mathrm{Spa}(A, A^\circ)|.$$

This realizes the Tate topos by a sober topological space, which is unique up to homeomorphism. In particular, there is an equivalence of categories of abelian sheaves

$$\mathrm{Sh}_T(\mathrm{MSpec}(A)) \cong \mathrm{Sh}(\mathcal{T}_A).$$

PROOF. Use the comparison of rational domains and their finite coverings from the preceding theorem. A sheaf on the rational Tate site defines a sheaf on the adic space by taking, on an arbitrary open, the inverse limit of its values on rational subsets contained in that open. Conversely, restriction to the rational basis gives a sheaf on the Tate site. The basis sheaf criterion shows that these operations are inverse. Since the strong Tate topology has the same sheaf category as its rational basis, this proves the claimed equivalence, for sets and hence for abelian groups.

For the uniqueness assertion, the subobjects of the terminal object in a sheaf topos recover the lattice of opens. A sober space is recovered from this lattice as its completely prime filters of opens. Thus the statement about uniqueness uses the topos of sheaves of sets; the displayed equivalence for abelian sheaves is a consequence. □

2.6. Analytic Spaces. The spaces in the last subsection were built out of Tate rings. It is often useful to know whether an adic space is locally of this form without having to specify a cover explicitly. We can characterize this in terms of its points. Let X be an adic space.

Definition 3.49. A point $x \in X$ is *analytic* if, in an affinoid neighborhood $\mathrm{Spa}(A, A^+)$, the support

$$\mathfrak{p}_x = \{a \in A : |a(x)| = 0\}$$

is not an open ideal of A. This condition is independent of the affinoid neighborhood [**Hub93**, §3]. We say that X is analytic if all its points are analytic.

For an analytic point, the topology on its residue field $K_x = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is induced by a nontrivial rank one coarsening of its valuation, and is nondiscrete; write $\widehat{K_x}$ for its completion. The original valuation can have higher rank, but defines the same field topology. One way to see the rank one coarsening is to choose a topologically nilpotent element whose value at x is nonzero; its positive powers are cofinal among neighborhoods of zero [**SW20**, §4.2]. At a nonanalytic point the residue field is given the discrete topology. This convention matters: a nontrivial valuation on a discrete field is continuous, but does not make the corresponding point analytic.

Remark 3.50. A topologically nilpotent unit cannot lie in the support of a valuation. This suggests that spectra of Tate rings should be analytic. It turns out that these give exactly the local models we need.

Lemma 3.51. *X is analytic if and only if it is covered by spectra of Tate rings.*

Proof. Suppose first that A is Tate, and choose a topologically nilpotent unit ϖ. If a valuation on A had open support, that support would contain ϖ^N for some N, since $\varpi^N \to 0$. This is impossible for a proper ideal because ϖ is a unit. Thus every point of $\mathrm{Spa}(A, A^+)$ is analytic.

Conversely, let x be an analytic point in an affinoid chart $\mathrm{Spa}(A, A^+)$. Choose a ring of definition A_0 and an ideal of definition $I = (f_1, \dots, f_r)$. Since the support of x is not open, at least one $f_i(x)$ is nonzero. Choose i with $|f_i(x)|$ maximal. Then x lies in the rational subset

$$U_i = \{y : |f_j(y)| \leqslant |f_i(y)| \neq 0 \text{ for every } j\}.$$

In its rational localization B_i, the element f_i is a unit. A ring of definition before completion is $A_0[f_1/f_i, \dots, f_r/f_i]$, and its ideal of definition generated by I is the principal ideal (f_i). Thus f_i remains topologically nilpotent after completion, and B_i is Tate. The structure sheaf restricted to a rational subset agrees with that of its rational localization, so these are affinoid adic neighborhoods. They cover the analytic locus, which is all of X under the hypothesis. □

CHAPTER 4

Perfectoid Spaces

We have spent some time developing a theory of analytic spaces that lets us work with rather large rings. We now get to the main reason for doing so: perfectoid spaces. There are two things about these spaces that will be especially useful to us. First, they allow us to pass back and forth between mixed characteristic and characteristic p. Second, their structure sheaves have very strong acyclicity properties, which we will use when we discuss the pro-étale site.

1. Motivation

The main goal of this chapter is to explain the tilting equivalence for perfectoid spaces and the almost purity theorem. We will prove the algebraic tilting correspondence using almost ring theory and deformation theory, and then explain how it extends to spaces. The analytic acyclicity and ramification results needed along the way are substantial theorems; we will state precisely where we use them.

To get a feel for the first property, let us see what happens to a field when we adjoin infinitely many roots of a uniformizer. This leads to a rather striking theorem of Fontaine and Wintenberger:

THEOREM 4.1 (Fontaine-Wintenberger [FW79]). *Let K and K' be the completions of the fields*

$$\mathbf{Q}_p(p^{1/p^\infty}) \quad \text{and} \quad \mathbf{F}_p((t^{1/p^\infty})),$$

respectively. The field-of-norms correspondence identifies their categories of finite separable extensions and hence their absolute Galois groups, up to the usual choice of compatible geometric basepoints.

What is remarkable here is that the two fields are of different characteristics, yet have the same theory of finite separable extensions. Somehow, after adding enough ramification, p has come to play much the same role in K as t does in K'.

The passage from K to K' is an instance of a much more general construction, which we call *tilting*.

DEFINITION 4.2. A perfectoid field is a complete nonarchimedean field K of residue characteristic p, whose topology is induced by a nondiscrete rank 1 valuation, such that Frobenius $\Phi\colon K^\circ/p \to K^\circ/p$ is surjective. Here $K^\circ = \mathcal{O}_K$ is its valuation ring. In characteristic p, this says that K is perfect.

DEFINITION 4.3. Let K be a perfectoid field. Its tilt $K^\flat$ is the fraction field of

$$K^{\flat\circ} = \varprojlim_{\Phi} K^\circ/p.$$

The valuation is given by $|x| = |x^\sharp|$, where, for $x = (x_0, x_1, \ldots)$ and arbitrary lifts $\widetilde{x}_n \in K^\circ$,

$$x^\sharp = \lim_{n\to\infty} \widetilde{x}_n^{p^n}.$$

In characteristic p, this recovers K itself. The elementary field construction, including convergence of this limit and completeness of $K^\flat$, is given in [**Sch12**, Proposition 3.6].

We fix a convention that will make the fractional powers below unambiguous. Choose $\varpi^\flat \in K^{\flat\circ}$ with $|p| \leqslant |\varpi^\flat| < 1$, put $\varpi = (\varpi^\flat)^\sharp$, and set

$$\varpi^{1/p^n} = \left((\varpi^\flat)^{1/p^n}\right)^\sharp.$$

Thus these are compatible actual roots in K, and $p \in \varpi K^\circ$. We use ϖ^r only for $r \in \mathbf{Z}[p^{-1}]$; an expression such as $|\varpi|^c$ with real c is only a numerical bound. The field construction gives

$$K^{\flat\circ}/\varpi^\flat \cong K^\circ/\varpi,$$

and the images of the two maximal ideals agree under this isomorphism. We are choosing a convenient pseudouniformizer, rather than assuming that an arbitrary one has compatible roots.

The theorem of Fontaine and Wintenberger is a special case of a much more general result of Scholze:

Theorem 4.4 (Field tilting, [**Sch12**, Theorem 3.7]). *Let K be a perfectoid field. The tilting functor $L \mapsto L^\flat$ induces an equivalence between the category of finite extensions of K and finite extensions of $K^\flat$.*

We use this field theorem as a ramification-theoretic input; its proof does not require the later almost purity theorem for general perfectoid algebras. In particular, after compatible choices of separable closures, one obtains an isomorphism between Gal_K and $\mathrm{Gal}_{K^\flat}$. To see why this is a remarkable statement, let us first consider some nonexamples, to see what can go wrong.

Example 4.5. The same inverse-limit expression can be formed outside the perfectoid setting, but it need not retain the field's information.

- Let $K = \mathbf{Q}_p$. Then K is discretely valued, so it is not perfectoid. Here $\varprojlim_\Phi K^\circ/p = \mathbf{F}_p$, and the proposed tilting equivalence drastically fails: $\mathrm{Gal}_{\mathbf{F}_p} \cong \hat{\mathbf{Z}}$ while $\mathrm{Gal}_{\mathbf{Q}_p}$ is far more complicated.
- Let $\ell \neq p$ be a prime and let $K = \mathbf{Q}_p\left(p^{1/\ell^\infty}\right)^\wedge$. Frobenius on K°/p is not surjective: a nonzero element of valuation $1/\ell^n < 1$ cannot have a pth root modulo p, since $1/(p\ell^n)$ is not in the value group $\mathbf{Z}[1/\ell]$. Again the inverse limit is just $\mathbf{F}_p$. Indeed, a nonconstant compatible sequence would give a positive valuation admitting division by every power of p in $\mathbf{Z}[1/\ell]$, which is impossible.

We would like to understand this phenomenon not just for fields but also for algebras. To get there, we first need a way of working with extensions whose ramification is arbitrarily small, as in the following example.

Example 4.6. Let $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ be a perfectoid field. Assume $p \neq 2$, so $L = K(\sqrt{p})$ is a degree 2 extension of K. Even though L/K is ramified, it is *almost unramified*! To make this precise, consider the finite level $K_n = \mathbf{Q}_p(p^{1/p^n})$ and its degree 2 extension $L_n = K_n(\sqrt{p})$. Then L_n° is generated by $p^{1/(2p^n)}$ over K_n°, and the relative different is $\mathcal{D} = \mathcal{D}_{L_n^\circ/K_n^\circ} = (p^{1/(2p^n)})$. Note that the additive valuation of $\mathcal{D}$ approaches 0 as $n \to \infty$—that is, the ramification becomes arbitrarily small!

Here is the precise differential calculation. Write $a_n = p^{1/p^n}$ and $b_n = p^{1/(2p^n)}$. Then

$$L_n^\circ = K_n^\circ[b_n], \qquad \Omega_{L_n^\circ/K_n^\circ} \cong L_n^\circ/(2b_n)\, db_n.$$

Since 2 is a unit, this module is killed by b_n, and in particular by a_n. This is a statement at level n; it does not say that a fixed finite-level module is killed by the maximal ideal at infinite level.

To pass to that level, put $A_\infty = \bigcup_n K_n^\circ$ and $B_\infty = \bigcup_n L_n^\circ$. Differentials commute with these filtered unions. Given a differential represented at a finite level and $\epsilon \in A_\infty$ of positive valuation, choose a later level m with $v(b_m) \leqslant v(\epsilon)$. Its image there is killed by b_m, hence by ϵ. Thus $\Omega_{B_\infty/A_\infty}$ is killed by the maximal ideal of A_∞. The corresponding assertion for the completed valuation rings will follow from almost purity; ordinary Kähler differentials should not simply be commuted with completion.

If we had a way of ignoring modules annihilated by $\mathfrak{m}$, the extension in this example would start to look unramified. This is the idea behind almost mathematics: we work in a category in which such modules are zero. We can then talk about *almost finite étale* algebras in this category. The almost purity theorem says, in particular, that if L/K is a finite separable extension, then L°/K° is almost finite étale.

2. Almost Commutative Algebra

Let K be a perfectoid field with valuation ring K° and maximal ideal $\mathfrak{m}$.

2.1. Preliminary Constructions.

Definition 4.7. Let M be a K°-module. An element $x \in M$ is *almost zero* if $\mathfrak{m}x = 0$. M is *almost zero* if every element of M is almost zero, or equivalently, $\mathfrak{m}M = 0$.

To treat the almost zero modules we encountered in Example 4.6 as actual zero objects, we would like a map with almost zero kernel and cokernel to become an isomorphism. This is precisely the sort of construction for which Serre quotients are useful.

Definition 4.8. Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* $\mathcal{B} \subset \mathcal{A}$ is a full subcategory of $\mathcal{A}$ satisfying the following property: Given any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$, we have $M \in \mathcal{B}$ if and only if $M', M'' \in \mathcal{B}$.

Given a Serre subcategory $\mathcal{B} \subset \mathcal{A}$ of an abelian category, we may form the quotient $\mathcal{A}/\mathcal{B}$, uniquely characterized by the following universal property:

Proposition 4.9. *Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory of an abelian category. Then there is an abelian category $\mathcal{A}/\mathcal{B}$ and an exact functor $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ such that F is essentially surjective, and $\ker F = \mathcal{B}$. Moreover, given any exact functor $G\colon \mathcal{A} \to \mathcal{C}$ such that $\mathcal{B} \subset \ker G$, there is an exact functor $H\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ and a natural isomorphism $G \cong H \circ F$. This factorization is unique up to a unique compatible natural isomorphism. We work in a universe in which the quotient Hom sets below are sets.*

Proof. The objects of the quotient are those of $\mathcal{A}$. Define

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) = \varinjlim_{\substack{M' \subset M,\ M/M' \in \mathcal{B} \\ N' \subset N,\ N' \in \mathcal{B}}} \operatorname{Hom}_{\mathcal{A}}(M', N/N').$$

The transition maps restrict the source and further quotient the target. The indexing system is filtered: intersect two choices of M' and add two choices of N', using the

Serre property. In particular, the allowed inclusion has *cokernel* in $\mathcal{B}$, while the allowed quotient has *kernel* in $\mathcal{B}$.

To compose two representatives, first restrict the source of the first map to the inverse image of the source of the second, modulo its discarded subobject. Then quotient the final target by the image of that discarded subobject. All the new discarded kernels and cokernels belong to $\mathcal{B}$, so this gives another allowed representative. Passing to a common refinement shows that the composite is independent of these choices and is associative. Addition is defined after taking common refinements as well.

An inclusion or quotient whose discarded term lies in $\mathcal{B}$ becomes invertible. Thus any representative $M' \to N/N'$ can be viewed as an actual morphism after replacing its source and target by isomorphic objects in the quotient. Its kernel and cokernel in $\mathcal{A}$ then represent its kernel and cokernel in the quotient. The usual identity between coimage and image in $\mathcal{A}$ proves that the quotient is abelian and that F is exact. The identity of M becomes zero precisely when some $M' \subset M$ with $M/M' \in \mathcal{B}$ is contained in an object of $\mathcal{B}$, which is equivalent to $M \in \mathcal{B}$.

Finally, an exact G as in the statement sends every morphism with kernel and cokernel in $\mathcal{B}$ to an isomorphism. It therefore sends the representative above to

$$G(M) \xrightarrow{\sim} G(M') \longrightarrow G(N/N') \xrightarrow{\sim} G(N).$$

This defines H, and the construction of kernels and cokernels proves its exactness. Since every quotient morphism has such a representative, both the factorization and its compatibility on morphisms are forced. This proves the asserted universal property. □

Proposition 4.10. *The full subcategory of K°-modules consisting of the almost zero K°-modules is a Serre subcategory.*

Proof. If M is annihilated by $\mathfrak{m}$, then any submodule or quotient of M is also annihilated by $\mathfrak{m}$. Conversely, if M' and M'' are annihilated by $\mathfrak{m}$, and M is an extension of M'' by M', then $\mathfrak{m}^2 M = 0$. But since the valuation on K is nondiscrete (since K is perfectoid), we have $\mathfrak{m}^2 = \mathfrak{m}$. □

We may therefore consider the quotient of the category of K°-modules by its full Serre subcategory of almost zero K°-modules.

Definition 4.11. The quotient category is called the category of *almost* K°-modules, denoted $\mathrm{Mod}(K^{\circ a})$.

For any $M \in \mathrm{Mod}(K^\circ)$, we write M^a for its image under the canonical functor $\mathrm{Mod}(K^\circ) \to \mathrm{Mod}(K^{\circ a})$. The objects are represented by the original modules, but their morphisms are now computed in the quotient category.

So far we have only talked about modules. In order to talk about algebras in the almost category, we also need a tensor product. Fortunately, the usual tensor product of K°-modules descends to the quotient.

Definition 4.12. Let M and N be K°-modules. We define

$$M^a \otimes_{K^{\circ a}} N^a := (M \otimes_{K^\circ} N)^a$$

This is well-defined on $\mathrm{Mod}(K^{\circ a})$, and gives $\mathrm{Mod}(K^{\circ a})$ the structure of an abelian tensor category.

Alternatively, we may describe

$$\mathrm{Hom}_{K^{\circ a}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N)$$

The Hom group on the right is an ordinary K°-module, rather than just an object of the almost category. We define the *almost morphisms* between M^a and N^a by

$$\mathrm{alHom}(M^a, N^a) = \mathrm{Hom}_{K^{\circ a}}(M^a, N^a)^a$$

Remark 4.13. There is a right adjoint to the canonical localization functor $\mathrm{Mod}(K^\circ) \to \mathrm{Mod}(K^{\circ a})$, called the functor of *almost elements*, given by sending

$$M \mapsto M_* := \mathrm{Hom}_{K^{\circ a}}(K^{\circ a}, M)$$

Moreover, we have isomorphisms $(M_*)^a \cong M$ and $(M^a)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M)$.

The functor of almost elements is quite useful because it takes us back to ordinary modules, where we can actually work with elements. Here are a few examples.

Example 4.14. Let K, K°, and $\mathfrak{m}$ be as above. Then:

- If M is an ordinary module annihilated by $\mathfrak{m}$, then $(M^a)_* = 0$.
- If M is a torsion-free K°-module, then we have

$$(M^a)_* \cong \{m \in M \otimes_{K^\circ} K : \epsilon \cdot m \in M \text{ for all } \epsilon \in \mathfrak{m}\}$$

2.2. Almost Properties. We can now begin to develop some familiar commutative algebra in the almost setting. One property worth looking at carefully is finite generation: recall that the integral extension in our motivating example was not even finitely generated in the usual sense. We will see that there is a useful almost version of this property.

Definition 4.15. Let A be a $K^{\circ a}$-algebra and M be an A-module.

- We say M is *flat* if the functor $X \mapsto X \otimes_A M$ is exact on A-modules.
- We say M is *almost projective* if the functor $X \mapsto \mathrm{alHom}_A(M, X)$ is exact on A-modules.
- Let R be a K°-algebra and N an R-module such that $M = N^a$ and $A = R^a$. Then we say M is *almost finitely generated* if for all $\epsilon \in \mathfrak{m}$, there is some finitely generated R-module N_ϵ, together with a map $f_\epsilon \colon N_\epsilon \to N$ such that both $\ker f_\epsilon$ and $\mathrm{coker}\, f_\epsilon$ are annihilated by ϵ. Moreover, we say M is *uniformly finitely generated* if there is some integer n such that N_ϵ can be generated by n elements for all ϵ.
- We say M is *almost finitely presented* if, for every nonzero $\epsilon, \delta \in \mathfrak{m}$, there is a complex $R^r \to R^s \to N$ with r, s finite such that ϵ kills the cokernel and δ kills the homology at R^s.

Example 4.16. Let $p \neq 2$ be a prime, and again consider $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$ and $L = K(\sqrt{p})$. Then L° is *not* finitely generated as a K°-module. To see this, write $L = K \oplus K\sqrt{p}$. The two summands have valuations in different cosets of $\mathbf{Z}[1/p]$, so their terms cannot cancel. In the second summand, integral elements have positive valuations tending to zero, but no element has valuation zero. A finitely generated K°-submodule would give a smallest valuation in that summand, which is impossible.

Nevertheless, let $b_n = p^{1/(2p^n)}$ and $M_n = K^\circ \cdot 1 + K^\circ \cdot b_n \subset L^\circ$. This is a free module of rank two. For any nonzero $\epsilon \in \mathfrak{m}$, choose n with $v(b_n) \leqslant v(\epsilon)$. The same decomposition

shows that $\epsilon L^\circ \subset M_n$. Thus $M_n \hookrightarrow L^\circ$ has zero kernel and cokernel killed by ϵ. In fact the maps $M_n \hookrightarrow L^\circ$ and $L^\circ \xrightarrow{\epsilon} M_n$ compose to multiplication by ϵ on both modules. These approximate splittings show that $L^{\circ a}$ is almost projective and almost finitely presented, with a uniform bound of two generators.

We finally have enough language to make precise what we meant by an extension being almost finite étale in the discussion of ramification.

Definition 4.17. Let A be a $K^{\circ a}$-algebra, and B an A-algebra.

- We say $A \to B$ is *almost unramified* if there is some element $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu(e) = 1$ and $eI_* = 0$, where $I = \ker(\mu : B \otimes_A B \to B)$ and μ is the multiplication morphism.
- We say $A \to B$ is étale if it is almost unramified and B is flat as an A-module.
- We say $A \to B$ is *finite étale* if it is étale and B is almost finitely presented as an A-module. We denote the category of finite étale A-algebras by $A_{\text{fét}}$.

Remark 4.18. The definition of almost unramified comes from algebraic geometry: For an affine morphism of finite type, being unramified is equivalent to its diagonal being an open immersion. The diagonal is also closed because the morphism is affine. Thus, in this situation, it is an open and closed immersion. In the above definition, I corresponds to the functions vanishing on the diagonal, and e is the characteristic function on the diagonal.

Remark 4.19. The idempotent e splits the diagonal: multiplication identifies $(B \otimes_A B)e$ with B. It also gives $I = (1-e)(B \otimes_A B)$, hence $I = I^2$ and $\Omega_{B/A} = I/I^2 = 0$ in the almost category. The converse cannot be asserted for arbitrary algebras merely from the vanishing of differentials. Finiteness hypotheses are needed to recover an idempotent from this infinitesimal condition, and we will use the idempotent definition above.

We call an almost module *finite projective* if it is almost finitely presented and almost projective. It is *uniformly finite projective* if, in addition, its almost finite generation has a uniform bound as above. These are almost properties, rather than a claim that its module of almost elements is an ordinary finite projective module.

There is another equivalent characterization of finite étale morphisms in terms of a trace morphism. Let A be any $K^{\circ a}$-algebra, and P some finite projective A-module. Then $P^* = \text{alHom}(P, A)$ is again finite projective. Moreover, we have a canonical isomorphism $P^{**} \cong P$ and a canonical isomorphism $\text{End}(P)^a \cong P \otimes_A P^*$. This gives a map $\text{tr}_{P/A} : \text{End}(P)^a \to A$.

Definition 4.20. Let A be a $K^{\circ a}$-algebra, and B an A-algebra such that B is finite projective as an A-module. The trace form is the bilinear form $t : B \otimes_A B \to A$ given by the composition of $\mu : B \otimes_A B \to B$ and the trace map $B \to A$, which sends any $b \in B$ to the trace of the endomorphism $b' \mapsto bb'$.

Remark 4.21. The definition of the endomorphism $b' \mapsto bb'$ does not literally make sense, as we cannot talk about some element b of an almost object B. What we really mean is as follows: One can define a map $B_* \to \text{End}_{A_*}(B_*)$ in the manner described, and then we consider the induced map on almost objects $B \to \text{End}_{A_*}(B_*)^a = \text{End}_A(B)^a$.

Theorem 4.22 ([GR03, Theorem 4.1.14]). *In the situation of Definition 4.20, the morphism $A \to B$ is finite étale if and only if the corresponding trace map is a perfect pairing.*

Recall that an ordinary finite étale algebra lifts uniquely across a nilpotent thickening. We will need the almost analogue of this fact, which allows us to recover an algebra over a complete integral ring from its reduction modulo ϖ.

THEOREM 4.23 (Almost formal lifting, [Sch12, Theorem 4.17]). *Let A be a $K^{\circ a}$-algebra. Assume that A is flat over $K^{\circ a}$ and ϖ-adically complete. Then the functor $B \mapsto B \otimes_A A/\varpi$ induces an equivalence of categories $A_{fét} \cong (A/\varpi)_{fét}$. Any $B \in A_{fét}$ is again flat over $K^{\circ a}$ and ϖ-adically complete. Finally, this functor preserves uniform finite projectivity.*

We use this as a theorem of almost ring theory. Its substantive input is [GR03, Theorem 5.3.27]: lifting through the square-zero extensions $A/\varpi^{n+1} \to A/\varpi^n$, followed by passage to the complete limit, is effective for almost finite étale algebras. Ordinary idempotent lifting alone would not justify this passage for almost modules.

Let us see how this helps with tilting. Rather than comparing the finite étale algebras on the two generic fibers directly, we can pass to the almost integral rings and then reduce modulo the pseudouniformizer. In other words, we would like to show that all the following categories are equivalent:

$$K_{\text{fét}} \cong K^{\circ a}_{\text{fét}} \cong (K^{\circ a}/\varpi)_{\text{fét}} \cong (K^{\flat \circ a}/\varpi^\flat)_{\text{fét}} \cong K^{\flat \circ a}_{\text{fét}} \cong K^\flat_{\text{fét}}$$

The middle identification $(K^{\circ a}/\varpi)_{\text{fét}} \cong (K^{\flat \circ a}/\varpi^\flat)_{\text{fét}}$ follows simply from the fact that $K^{\circ a}/\varpi \cong K^{\flat \circ a}/\varpi^\flat$. Theorem 4.23 gives the inner two equivalences: $K^{\circ a}_{\text{fét}} \cong (K^{\circ a}/\varpi)_{\text{fét}}$ and $(K^{\flat \circ a}/\varpi^\flat)_{\text{fét}} \cong K^{\flat \circ a}_{\text{fét}}$. For the remaining two outer equivalences, it is more convenient to discuss the theory in the language of *perfectoid K-algebras*, which we now introduce.

3. Perfectoid Algebras

DEFINITION 4.24. Throughout, fix a perfectoid field K, with tilt $K^\flat$.

- A perfectoid K-algebra is a Banach K-algebra R such that the subset of power-bounded elements $R^\circ \subset R$ is open and bounded, and the Frobenius morphism $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi$ is surjective. Morphisms of perfectoid K-algebras are the continuous morphisms of K-algebras.
- A perfectoid $K^{\circ a}$-algebra is a ϖ-adically complete flat $K^{\circ a}$-algebra A on which Frobenius induces an isomorphism $\Phi\colon A/\varpi^{1/p} \cong A/\varpi$. Morphisms of perfectoid $K^{\circ a}$-algebras are the morphisms of $K^{\circ a}$-algebras.
- A perfectoid $K^{\circ a}/\varpi$-algebra is a flat $K^{\circ a}/\varpi$-algebra $\bar{A}$ such that Frobenius induces an isomorphism $\Phi\colon \bar{A}/\varpi^{1/p} \cong \bar{A}$. Morphisms are morphisms of $K^{\circ a}/\varpi$-algebras.

Write $\mathrm{Perf}(K)$ for the category of perfectoid K-algebras. We use analogous notation for the other two categories.

The tilting equivalence upgrades to an equivalence $\mathrm{Perf}(K) \cong \mathrm{Perf}(K^\flat)$. In particular, we have the following chain of equivalences:

THEOREM 4.25. *We have the following chain of equivalences of categories:*

$$\mathrm{Perf}(K) \cong \mathrm{Perf}(K^{\circ a}) \cong \mathrm{Perf}(K^{\circ a}/\varpi) \cong \mathrm{Perf}(K^{\flat \circ a}/\varpi^\flat) \cong \mathrm{Perf}(K^{\flat \circ a}) \cong \mathrm{Perf}(K^\flat)$$

The point of expressing the theorem this way is that we can forget a surprising amount of information about a perfectoid algebra without losing the algebra itself. Passing first to its almost integral model and then reducing modulo ϖ is reversible. And once we have reduced modulo ϖ, there is no longer a difference between the two characteristics.

Let us start with the equivalence between $\mathrm{Perf}(K^{\circ a})$ and $\mathrm{Perf}(K)$. For this, it will be useful to know when flatness and completeness can be checked on the module of almost elements.

LEMMA 4.26. *Let M be a $K^{\circ a}$-module.*

(i) *The following are equivalent:*
- *M is flat over $K^{\circ a}$.*
- *M_* is flat over K°.*
- *M_* is ϖ-torsion-free.*

(ii) *If N is a flat K°-module and $M = N^a$, then M is flat over $K^{\circ a}$ and*
$$M_* = \{x \in N[\varpi^{-1}] : \epsilon x \in N \text{ for all } \epsilon \in \mathfrak{m}\}$$

(iii) *If M is flat over $K^{\circ a}$, then for all $x \in K^\circ$, we have $(xM)_* = xM_*$. Moreover, $M_*/xM_* \hookrightarrow (M/xM)_*$, and for every nonzero $\epsilon \in \mathfrak{m}$ the image of*
$$(M/x\epsilon M)_* \longrightarrow (M/xM)_*$$
is exactly M_*/xM_*.

(iv) *If M is flat over $K^{\circ a}$, then M is ϖ-adically complete if and only if M_* is ϖ-adically complete.*

PROOF. We give the details, following the module calculation of [**Sch12**, Lemma 5.3].

For (i), an ordinary module over the valuation ring K° is flat if and only if it is torsion-free. Since every nonzero element of K° divides a sufficiently large power of ϖ, this is equivalent to having no ϖ-torsion. If M_* is flat, its localization $M = (M_*)^a$ is flat. Conversely, flatness of M makes the kernel of multiplication by ϖ on M_* almost zero. A module of almost elements has no nonzero almost zero element: if a map $K^{\circ a} \to M$ is killed by $\mathfrak{m}$, its image is zero, since $\mathfrak{m}^2 = \mathfrak{m}$. Thus that kernel is zero.

For (ii), use $(N^a)_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N)$. Tensoring such a homomorphism with K identifies it with multiplication by an element of $N[\varpi^{-1}]$, because $\mathfrak{m} \otimes_{K^\circ} K = K$. It takes values in N exactly when all its ϵ-multiples do. This proves the formula, and flatness follows from exactness of localization.

For (iii), the formula in (ii) applied to xM_* shows that xM_* is already saturated, meaning that an element belongs to it whenever all its $\mathfrak{m}$-multiples do. Hence $(xM)_* = xM_*$. Left exactness of $(-)_*$ gives the displayed inclusion. To check the image assertion, let $u \in (M/x\epsilon M)_*$, represented by a homomorphism
$$\mathfrak{m} \longrightarrow M_*/x\epsilon M_*.$$
Call this homomorphism f, and lift $f(\epsilon)$ to $v \in M_*$. For every $\delta \in \mathfrak{m}$, linearity gives $\delta v \in \epsilon M_*$. Saturation of ϵM_* therefore gives $v = \epsilon w$ for some $w \in M_*$. If $\widetilde{f(\delta)} \in M_*$ lifts $f(\delta)$, linearity now says
$$\delta\epsilon w \equiv \epsilon \widetilde{f(\delta)} \pmod{x\epsilon M_*}.$$
Torsion-freeness lets us cancel ϵ, giving $\delta w \equiv \widetilde{f(\delta)} \pmod{xM_*}$. Thus the image of u in $(M/xM)_*$ is exactly the image of w. The reverse inclusion is obtained by lifting w itself.

Finally, both $(-)_*$ and localization commute with inverse limits. The first is a right adjoint; the second has the left adjoint $M \mapsto \mathfrak{m} \otimes_{K^\circ} M_*$. Part (iii), with $x = \varpi^n$ and $\epsilon = \varpi$, gives
$$\varprojlim_n (M/\varpi^n M)_* = \varprojlim_n M_*/\varpi^n M_*.$$

Applying the two functors to the completeness map proves (iv). □

Corollary 4.27. *Let R be a perfectoid K-algebra. Then the Frobenius Φ induces an isomorphism $R^\circ/\varpi^{1/p} \cong R^\circ/\varpi$, and $R^{\circ a}$ is a perfectoid $K^{\circ a}$-algebra.*

Proof. Since R is perfectoid, by definition Φ is surjective. If $x \in R^\circ$ is such that x^p/ϖ is powerbounded, then $x/\varpi^{1/p}$ is also powerbounded, so Φ is injective as well. Finally R° is ϖ-adically complete and flat over K°, so the previous lemma gives that $R^{\circ a}$ is ϖ-adically complete and flat over $K^{\circ a}$. □

Lemma 4.28. *Let A be a perfectoid $K^{\circ a}$-algebra, and $R = A_*[\varpi^{-1}]$. Equip R with the Banach K-algebra structure so that A_* is open and bounded. Then $R^\circ = A_*$ is the set of powerbounded elements, R is perfectoid, and there is an isomorphism $\Phi\colon A_*/\varpi^{1/p} \cong A_*/\varpi$.*

Before beginning the proof, note that the almost isomorphism

$$\Phi\colon A_*/\varpi^{1/p} \longrightarrow A_*/\varpi$$

is actually injective. If $y^p \in \varpi A_*$, almost injectivity says that $\epsilon y \in \varpi^{1/p} A_*$ for every $\epsilon \in \mathfrak{m}$. Lemma 4.26(iii), or its saturation description, then gives $y \in \varpi^{1/p} A_*$. We will use this in the following auxiliary result.

Lemma 4.29. *Suppose $x \in R$ satisfies $x^p \in A_*$. Then $x \in A_*$.*

Proof. Since Φ is injective, if $y \in A_*$ satisfies $y^p \in \varpi A_*$, then $y \in \varpi^{1/p} A_*$. There is always some $k \geqslant 0$ such that $y = \varpi^{k/p} x \in A_*$. As long as $k \geqslant 1$, we have that $y^p \in \varpi A_*$, so that $y \in \varpi^{1/p} A_*$. Since A_* has no ϖ-torsion, we see that $\varpi^{(k-1)/p} x \in A_*$. By induction, we see that $x \in A_*$. □

We can now use this observation to finish the proof of Lemma 4.28: we need to recover the integral subring from the generic fiber.

Proof. Assume that $x \in R$ is powerbounded. Then ϵx is topologically nilpotent for all $\epsilon \in \mathfrak{m}$. Thus for all N sufficiently large, we have $(\epsilon x)^{p^N} \in A_*$. But then $\epsilon x \in A_*$ by the previous lemma. Since this is true for all $\epsilon \in \mathfrak{m}$, we have $x \in A_*$ by Lemma 4.26 (ii).

Next, we show that Φ is surjective. Its cokernel C is annihilated by $\mathfrak{m}$, so it suffices to show that every class modulo ϖ is a Frobenius image modulo $\mathfrak{m}A_*$: the remaining error then maps to $\mathfrak{m}C = 0$. Let $x \in A_*$ be arbitrary. By almost surjectivity, there is some $y \in A_*$ such that $\varpi^{1/p} x \equiv y^p$ modulo ϖA_*. Letting $z := \frac{y}{\varpi^{1/p^2}}$, we see that $z^p \equiv x$ modulo $\varpi^{(p-1)/p} A_*$, and so $z^p \in A_*$, and thus $z \in A_*$ also. Finally, since $x \equiv z^p$ modulo $\mathfrak{m}A_*$, we see the desired surjectivity.

Putting it all together, R is a Banach K-algebra such that R° is open and bounded, and such that Φ is surjective on A_*/ϖ, which by definition means that R is perfectoid. □

This shows that passing to the almost integral model and then inverting the pseudouniformizer (with the almost elements taken first) are inverse constructions. We have proved the first equivalence.

Corollary 4.30. *There is an equivalence* $\mathrm{Perf}(K^{\circ a}) \cong \mathrm{Perf}(K)$.

Proof. The two functors are $A \mapsto A_*[\varpi^{-1}]$ and $R \mapsto R^{\circ a}$. Lemma 4.28 shows that their composite on almost algebras is the identity. For the other composite, it remains to observe that $(R^{\circ a})_* = R^\circ$. Indeed, if $\epsilon x \in R^\circ$ for every $\epsilon \in \mathfrak{m}$, choose, for each n, a

nonzero $\epsilon_n \in \mathfrak{m}$ with $|\epsilon_n|^n \geqslant |\varpi|$. Then $x^n \in \varpi^{-1}R^\circ$ for all n, so x is powerbounded. Such ϵ_n exist because the valuation is nondiscrete. All the constructions commute with morphisms: continuous maps send bounded sets, hence powerbounded elements, to bounded sets, and a map of the integral almost algebras gives a continuous map after inverting ϖ. Thus these are equivalences of categories, not just correspondences on objects. □

The only real difficulty left is to see why an almost integral perfectoid algebra is determined by its reduction modulo ϖ. Proving this also takes care of the corresponding equivalence on the tilted side of Theorem 4.25.

THEOREM 4.31. *The functor $A \mapsto A/\varpi$ gives an equivalence of categories* $\mathrm{Perf}(K^{\circ a}) \cong \mathrm{Perf}(K^{\circ a}/\varpi)$.

Thus, we need to recover an algebra from its reduction modulo ϖ. It is helpful to do this one power of ϖ at a time, lifting an algebra over $K^{\circ a}/\varpi^n$ to one over $K^{\circ a}/\varpi^{n+1}$. We must also lift maps between algebras, since our goal is an equivalence of categories.

For an étale algebra, this sort of lifting is governed by the familiar fact that an invertible derivative gives a unique correction to a root. For example, if $I^2 = 0$, $f(x) \in I$, and $f'(x)$ is a unit, then $x - f'(x)^{-1}f(x)$ is the unique root lifting x modulo I: the Taylor expansion has no terms of degree two in I. Perfectoid algebras are usually not finitely presented, so we need a version of deformation theory that also keeps track of relations between the defining equations. This is what the cotangent complex does.

4. The Cotangent Complex

The main reference for this section is [**GR03**]. Let $R \to S$ be a ring homomorphism. The cotangent complex $\mathbf{L}_{S/R} \in D_{\leqslant 0}(S)$ records the relations between differentials as well as the differentials themselves. One construction is to resolve S by a simplicial polynomial R-algebra $P_\bullet$, form $\Omega^1_{P_\bullet/R} \otimes_{P_\bullet} S$, and take its associated complex. Its degree-zero cohomology is $\Omega^1_{S/R}$.

For $R \to S \to T$, there is a transitivity triangle

$$T \otimes^{\mathbf{L}}_S \mathbf{L}_{S/R} \longrightarrow \mathbf{L}_{T/R} \longrightarrow \mathbf{L}_{T/S} \longrightarrow (T \otimes^{\mathbf{L}}_S \mathbf{L}_{S/R})[1].$$

There is also a derived base-change isomorphism: if $S' = S \otimes^{\mathbf{L}}_R R'$ is an ordinary ring, then

$$\mathbf{L}_{S'/R'} \cong \mathbf{L}_{S/R} \otimes^{\mathbf{L}}_S S'.$$

Flatness of S over R is one sufficient hypothesis for the derived tensor product to be ordinary. We will need this distinction in the proof below.

Let $I \subset R$ satisfy $I^2 = 0$, put $R_0 = R/I$, and fix a flat R_0-algebra S_0. The following are the deformation-theoretic inputs; we use them in their full form, including uniqueness, as in [**GR03**, Propositions 3.2.9 and 3.2.16] and [**Sch12**, Theorems 5.11 and 5.12].

THEOREM 4.32 (Deformations across a square-zero ideal). *There is an obstruction in*

$$\mathrm{Ext}^2_{S_0}(\mathbf{L}_{S_0/R_0}, S_0 \otimes_{R_0} I)$$

to finding a flat R-algebra S together with an identification $S \otimes_R R_0 \cong S_0$. Its vanishing is necessary and sufficient. When lifts exist, their isomorphism classes form a torsor under the

corresponding Ext^1 *group, and the automorphisms of a fixed lift inducing the identity on* S_0 *form the corresponding* $\mathrm{Ext}^0 = \mathrm{Hom}$ *group.*

THEOREM 4.33 (Lifting morphisms). *Let* S, S' *be flat* R*-algebras, with reductions* S_0, S_0'*, and let* $f_0 \colon S_0 \to S_0'$ *be an* R_0*-algebra map. There is an obstruction in*

$$\mathrm{Ext}^1_{S_0}(\mathbf{L}_{S_0/R_0}, S_0' \otimes_{R_0} I)$$

to lifting f_0*. When it vanishes, the possible lifts form a torsor under* $\mathrm{Hom}_{S_0}(\mathbf{L}_{S_0/R_0}, S_0' \otimes_{R_0} I)$*, where the coefficient module is viewed through* f_0*.*

These are general results of deformation theory, rather than claims that every flat algebra has a unique deformation. In our situation, all the relevant groups will vanish because the entire cotangent complex vanishes.

PROPOSITION 4.34. (i) *If* R *is a perfect* $\mathbf{F}_p$*-algebra, then* $\mathbf{L}_{R/\mathbf{F}_p} = 0$*.*

(ii) *Let* $R \to S$ *be any morphism of* $\mathbf{F}_p$*-algebras. Write* $R_{(\Phi)}$ *for* R *with scalar structure given by Frobenius. If the relative Frobenius is an isomorphism*

$$R_{(\Phi)} \otimes^{\mathbf{L}}_R S \longrightarrow S_{(\Phi)},$$

then $\mathbf{L}_{S/R} = 0$*.*

PROOF. We prove the second assertion first. Choose a simplicial polynomial resolution $P_\bullet \to S$ over R. The relative Frobenius on this resolution is represented by sending each polynomial generator T to T^p. Its map on relative differentials is zero, because $d(T^p) = 0$ in characteristic p. Thus its induced map on cotangent complexes is zero as well; this is a statement about the whole resolution, not only about $\Omega^1_{S/R}$.

On the other hand, the assumed derived isomorphism and derived base change identify this same map with an isomorphism onto the cotangent complex of the target. A zero morphism can be an isomorphism only when its target is zero. Removing the Frobenius notation on the scalar structures gives $\mathbf{L}_{S/R} = 0$. For (i), apply (ii) to $\mathbf{F}_p \to R$: Frobenius on the base is the identity, while Frobenius on the perfect algebra R is an isomorphism. □

Gabber and Ramero construct the same theory in the almost category. If $R \to S$ is a map of ordinary K°-algebras, then $\mathbf{L}^a_{S/R}$ depends only on $R^a \to S^a$. We write $\mathbf{L}^a_{B/A}$ for the resulting complex of almost modules. Transitivity, derived base change, the two deformation theorems, and the Frobenius argument remain valid there; this is the almost cotangent-complex theory of [**GR03**, Section 3.2].

COROLLARY 4.35. *Let* $\bar{A}$ *be a perfectoid* $K^{\circ a}/\varpi$*-algebra. Then*

$$\mathbf{L}^a_{\bar{A}/(K^{\circ a}/\varpi)} = 0.$$

PROOF. Put $R_0 = K^{\circ a}/\varpi$. It has characteristic p, and Frobenius on it is surjective with kernel generated by $\varpi^{1/p}$. Since $\bar{A}$ is flat over R_0, the derived relative Frobenius is the ordinary relative Frobenius. Identifying its source by this description of the kernel gives exactly the isomorphism

$$\Phi \colon \bar{A}/\varpi^{1/p} \xrightarrow{\sim} \bar{A}$$

from Definition 4.24. Proposition 4.34, in the almost category, therefore applies. □

We now have the ingredients for Theorem 4.31. The proof has two parts. First, vanishing of the cotangent complex lets us lift the algebra and all its maps through the successive square-zero thickenings. We then take the inverse limit and check that its reductions are the algebras we constructed. The second step needs a completeness statement; it is useful to keep it separate from the lifting argument.

Proof of Theorem 4.31. Write $R_n = (K^\circ/\varpi^n)^a$ and start with $A_1 = \bar{A}$. We first check a useful fact: if A_n is any flat R_n-algebra lifting $\bar{A}$, then

$$L_n := \mathbf{L}^a_{A_n/R_n} = 0.$$

Indeed, flat derived base change gives

$$L_n \otimes^{\mathbf{L}}_{A_n} \bar{A} \cong \mathbf{L}^a_{\bar{A}/R_1} = 0.$$

The finite filtration of A_n by powers of ϖ has successive quotients isomorphic to $\bar{A}$, by flatness over R_n. Tensoring its short exact sequences with L_n therefore shows, one step at a time, that $L_n \otimes^{\mathbf{L}}_{A_n} A_n = 0$. This proves the claim. In particular, the argument uses $\mathbf{L}^a_{\bar{A}/R_1}$, not $\mathbf{L}^a_{\bar{A}/R_n}$; changing this base would change the complex.

Now $R_{n+1} \to R_n$ has square-zero kernel $(\varpi^n)/(\varpi^{n+1})$. All the Ext^2, Ext^1, and Hom groups in the deformation theorems vanish because $L_n = 0$. Hence A_n lifts to a flat A_{n+1}, uniquely up to a unique isomorphism inducing the chosen identification modulo ϖ^n. The same argument shows that every morphism between two such reductions lifts uniquely. Induction gives a compatible system (A_n) and shows that reduction to level one is fully faithful on these systems.

Set $A = \varprojlim_n A_n$ in the almost category. For this passage to the limit, we use the flat formal-completion fact in the final paragraph of the proof of [**Sch12**, Theorem 5.10]: compatible flat modules over $K^{\circ a}/\varpi^n$, whose transition maps are reduction modulo ϖ^n, have a flat ϖ-adically complete inverse limit, and its reduction modulo ϖ^n is the prescribed module. No finite-generation hypothesis is needed. The flatness hypothesis supplies the exact reduction sequences

$$0 \longrightarrow A_m \xrightarrow{\varpi^n} A_{m+n} \longrightarrow A_n \longrightarrow 0$$

for all m, n. The formal-completion fact is what allows one to recover these reductions from the limit; exactness of an arbitrary inverse limit would not suffice. Here flat formal completion supplies both the flatness of the limit and its prescribed reductions, in the almost form just cited.

Consequently $A/\varpi^n = A_n$, and in particular $A/\varpi = \bar{A}$. The Frobenius condition for A is the condition already satisfied by $\bar{A}$, since $A/\varpi^{1/p} = \bar{A}/\varpi^{1/p}$. Thus A is perfectoid. This proves essential surjectivity. If A and B are perfectoid lifts, the unique morphisms $A/\varpi^n \to B/\varpi^n$ lifting a given map modulo ϖ are compatible; completeness identifies their inverse limit with a unique map $A \to B$. This proves full faithfulness, and hence the equivalence. □

Theorem 4.25 now follows: apply this equivalence over both K and $K^\flat$, identify the middle reduction rings using $K^\circ/\varpi \cong K^{\flat\circ}/\varpi^\flat$, and use the already proved equivalence between the integral and Banach categories at the two ends.

5. Perfectoid Spaces

So far, we have been talking about perfectoid algebras. Let us put these algebras into the language of adic geometry, so that we can glue them to obtain more general spaces. Our ultimate reason for doing this is that suitable rigid spaces become perfectoid after passing to sufficiently fine pro-étale covers. We will make sense of this statement in the next chapter.

Let us first make the algebraic tilting functor concrete. The inverse-limit description associated to Theorem 4.25 is

$$R^{\flat\circ} \cong \varprojlim_{\Phi} R^\circ/\varpi, \qquad R^\flat = R^{\flat\circ}[(\varpi^\flat)^{-1}].$$

This agrees with the abstract equivalence above; see [**Sch12**, Proposition 5.17].

Definition 4.36. For $x = (x_0, x_1, \ldots) \in R^{\flat\circ}$, choose lifts $\widetilde{x}_n \in R^\circ$ of its coordinates and put

$$x^\sharp = \lim_{n\to\infty} \widetilde{x}_n^{p^n}.$$

Extend this to $R^\flat$ by $(x/(\varpi^\flat)^m)^\sharp = x^\sharp/\varpi^m$. This gives the multiplicative map $(-)^\sharp \colon R^\flat \to R$.

Notice that the coordinates live in the *integral* ring R°/ϖ, not in R/p. In mixed characteristic, p is invertible in R, so the latter quotient would be zero! To check the limit, use $p \in \varpi K^\circ$ and the binomial theorem: if two integral elements agree modulo ϖ^k, their pth powers agree modulo ϖ^{k+1}. Consecutive terms in the displayed limit therefore agree modulo ϖ^{n+1}. The same estimate proves independence of the lifts, and multiplication commutes with the limit. Addition need not commute with $\sharp$, although it does after reduction modulo ϖ.

Definition 4.37. A perfectoid affinoid K-algebra is a Huber pair (R, R^+) with R perfectoid and R^+ an open integrally closed subring of R°. Morphisms are continuous maps preserving the indicated subrings. In this chapter our pairs are over (K, K°), so $K^\circ \subset R^+$.

Lemma 4.38. *The categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras are equivalent. Under this correspondence,*

$$R^{\flat+} = \varprojlim_{x\mapsto x^p} R^+, \qquad R^{\flat+}/\varpi^\flat \cong R^+/\varpi.$$

The inverse limit in the first formula describes multiplicative monoids, with the addition inherited from the tilt.

Proof. Every topologically nilpotent $x \in R^\circ$ lies in R^+: some power x^n lies in the open subring R^+, and x is then integral over R^+. In particular $\mathfrak{m}R^\circ \subset R^+$. Consequently an open integrally closed subring $R^+ \subset R^\circ$ is determined by its image in $R^\circ/\mathfrak{m}R^\circ$. Conversely, the inverse image of an integrally closed subring of this quotient is open and integrally closed. To see the converse integrality assertion, reduce a monic equation and then lift it; the error in its constant term lies in $\mathfrak{m}R^\circ$ and already belongs to the inverse-image subring.

Reduction modulo ϖ followed by reduction modulo the maximal ideal gives a canonical isomorphism

$$R^{\flat\circ}/\mathfrak{m}^\flat R^{\flat\circ} \cong R^\circ/\mathfrak{m}R^\circ.$$

Transporting the subrings along this isomorphism defines $R^{\flat+}$ and its inverse construction. These operations commute with morphisms, so Theorem 4.25 gives the claimed equivalence for pairs. The inverse-limit and reduction formulas follow by applying the same construction to R^+; the multiplicative coordinates are $x^\sharp, (x^{1/p})^\sharp, \dots$. □

THEOREM 4.39. *Let (R, R^+) be perfectoid, and put*

$$X = \operatorname{Spa}(R, R^+), \qquad X^\flat = \operatorname{Spa}(R^\flat, R^{\flat+}).$$

(i) *The map $x \mapsto x^\flat$, defined by $|f(x^\flat)| = |f^\sharp(x)|$, is a homeomorphism identifying rational subsets.*
(ii) *If $U \subset X$ is rational, its affinoid pair is perfectoid, and*

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U))^\flat \cong (\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)).$$

(iii) *The presheaves $\mathcal{O}_X$ and $\mathcal{O}_{X^\flat}$ are sheaves.*
(iv) *$\mathfrak{m}H^i(X, \mathcal{O}_X^+) = 0$ for every $i > 0$.*

This is the affinoid tilting and acyclicity theorem of [**Sch12**, Theorem 6.3]. We will explain the rational-localization and topological arguments, including why the map is a homeomorphism. The almost acyclicity assertion in (iii)–(iv) will be used as an external analytic theorem. It is stronger than the sheaf property and does not follow merely from continuity of the tilting map.

PROPOSITION 4.40. *Let*

$$U = U\left(\frac{f_1, \dots, f_n}{g}\right) \subset \operatorname{Spa}(R^\flat, R^{\flat+})$$

be rational. After rescaling and adding a redundant numerator, we may assume $f_i, g \in R^{\flat\circ}$ and $f_n = (\varpi^\flat)^N$. Write $U^\sharp$ for its inverse image in $\operatorname{Spa}(R, R^+)$. Let D be the ϖ-adic completion of

$$D_0 = R^\circ\left[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right] \subset R[1/g^\sharp],$$

where the displayed roots are the sharp images of the chosen roots in the tilt. Then D^a is perfectoid,

$$\mathcal{O}_X(U^\sharp) = D[1/\varpi], \qquad \mathcal{O}_X(U^\sharp)^{\circ a} = D^a,$$

and the tilt of this algebra is $\mathcal{O}_{X^\flat}(U)$.

PROOF. First suppose that K has characteristic p. Put

$$P = R^\circ[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}], \qquad I = (g^{1/p^m} T_i^{1/p^m} - f_i^{1/p^m})_{i,m}.$$

The map $P/I \to D_0$ is surjective and becomes an isomorphism after inverting ϖ: the relation with $f_n = \varpi^N$ then makes g invertible. Both P and the system of relations are stable under taking pth roots, so P/I is perfect. If z in its kernel is killed by ϖ^k, then $(\varpi^{k/p^m} z)^{p^m} = 0$, and reducedness gives $\varpi^{k/p^m} z = 0$ for every m. Its kernel is therefore almost zero. Frobenius consequently induces an almost isomorphism $D_0/\varpi^{1/p} \to D_0/\varpi$. The ring D_0 is torsion-free over K°, and its completion is still torsion-free. Completion preserves these reductions, so D^a is perfectoid.

To identify the topology, let $E_0 = R^\circ[f_1^\sharp/g^\sharp, \dots, f_n^\sharp/g^\sharp]$. There are inclusions

$$E_0 \subset D_0 \subset \varpi^{-nN} E_0.$$

For the second, split each exponent of a monomial into its integer and fractional parts. At most one extra factor of $1/g^\sharp$ per variable is needed to absorb the fractional parts into R°, and $\varpi^N/g^\sharp \in E_0$. Thus the two rings define the same topology after inverting ϖ. Their completions give $\mathcal{O}_X(U^\sharp) = D[1/\varpi]$, and Lemma 4.28 gives the assertion about the almost integral ring whenever D^a is perfectoid.

For general K, apply the characteristic-p construction to U and let S be the untilt of $\mathcal{O}_{X^\flat}(U)$. The universal property of rational localization gives $\mathcal{O}_X(U^\sharp) \to S$, and all the chosen roots map into S°. Use the same polynomial presentation P/I, with the coefficients replaced by their sharp images. Modulo ϖ this presentation is identified with its characteristic-p counterpart. The composite

$$(P/(I,\varpi))^a \longrightarrow (D/\varpi)^a \longrightarrow S^{\circ a}/\varpi$$

is therefore an isomorphism, while its first map is surjective. Both maps are isomorphisms. The same comparison modulo $\varpi^{1/p}$ proves the Frobenius condition for D^a; flatness and completeness were already built into the construction. Now the topology calculation and Lemma 4.28 apply, and Theorem 4.31 identifies D^a with $S^{\circ a}$. This proves all the assertions. This is the algebraic content of [**Sch12**, Lemma 6.4]. □

The next result explains how to compensate for the failure of $\sharp$ to be additive. It is particularly useful when transferring equations between the two characteristics.

Proposition 4.41. *Let $R = K\langle T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle$ and let $f \in R^\circ$ be homogeneous of degree $d \in \mathbf{Z}[p^{-1}]$. For every rational $c \geqslant 0$ and every $\epsilon > 0$, there is a homogeneous $g \in R^{\flat\circ}$ of degree d such that, for all $x \in \operatorname{Spa}(R, R^\circ)$,*

$$|f(x) - g^\sharp(x)| \leqslant |\varpi|^{1-\epsilon} \max(|f(x)|, |\varpi|^c).$$

Proof. We give the approximation argument of [**Sch12**, Lemma 6.5]. We may replace ϵ by a smaller element of $\mathbf{Z}[p^{-1}] \cap (0,1)$. It suffices to obtain the estimate for a value of c at least as large as the one prescribed. If $d = 0$, then f is a scalar. Match its valuation by a sharp image in K and lift the resulting unit modulo ϖ; this gives the stronger relative error bound $|\varpi||f|$. The zero element is immediate. Thus assume $d > 0$.

Choose $a \in \mathbf{Z}[p^{-1}]$ with $0 < a < \epsilon$. We induct on $c = 0, a, 2a, \dots$, retaining a small positive reserve η_c in the estimate

$$|f - g_c^\sharp| \leqslant |\varpi|^{1-\epsilon+\eta_c} \max(|f|, |\varpi|^c).$$

At $c = 0$, lift the homogeneous reduction of f modulo ϖ and take $\eta_0 = \epsilon$. At each subsequent step we may decrease η_c, so assume $0 < \eta_c \leqslant \epsilon - a$.

Put $q = 1 - \epsilon + \eta_c$ and $h = f - g_c^\sharp$. On the rational set $|g_c| \leqslant |\varpi^\flat|^c$, whose inverse image is $|f| \leqslant |\varpi|^c$, we have $|h| \leqslant |\varpi|^{c+q}$. Proposition 4.40 describes its integral ring, up to almost isomorphism, by adjoining all roots of $g_c^\sharp/\varpi^c$. Choose $0 < \eta_{c+a} < \eta_c$ in $\mathbf{Z}[p^{-1}]$ and put $q' = 1 - \epsilon + \eta_{c+a}$. Absorbing the almost error into the factor $\varpi^{\eta_c - \eta_{c+a}}$, and taking the homogeneous degree-d part, gives a convergent expression

$$h = \sum_{i \in \mathbf{Z}[p^{-1}] \cap [0,1]} \varpi^{c+q'} (g_c^\sharp/\varpi^c)^i r_i, \qquad \deg r_i = d(1-i), \quad r_i \in R^\circ, \quad r_i \to 0.$$

There are no terms of degree $i > 1$, because $d > 0$ and all variable degrees are non-negative. Taking homogeneous parts is legitimate in the completed Tate algebra, whose monomials have a unique convergent expansion.

Choose homogeneous $s_i \in R^{\flat\circ}$ with $s_i \to 0$ and $r_i \equiv s_i^\sharp \pmod{\varpi}$, and define

$$g_{c+a} = g_c + \sum_i (\varpi^\flat)^{c+q'} \left(g_c/(\varpi^\flat)^c\right)^i s_i.$$

All scalar exponents $c(1-i)+q'$ are nonnegative, so this is an integral homogeneous element of degree d.

We check the new bound in two regions. If $|f(x)| > |\varpi|^c$, then $|g_c(x^\flat)| = |f(x)|$, and each correction term has size at most $|\varpi|^{q'}|f(x)|$: the largest possible bound occurs at $i = 1$. Reduction modulo ϖ, or the binomial estimate defining $\sharp$, gives the same bound for the change in sharp images; its additional error is at most $|\varpi||f(x)|$, which is smaller since $q' < 1$. Together with the inductive estimate, this gives the desired bound there.

On the other region, divide the defining expression by ϖ^c and reduce in the integral rational localization modulo ϖ. Sharp is additive after this reduction, and $r_i \equiv s_i^\sharp$, so

$$f - g_{c+a}^\sharp \in \varpi^{c+1}\mathcal{O}_X(|f| \leqslant |\varpi|^c)^\circ.$$

Since $c+1 \geqslant c+a+q'$, this is the required bound on that region as well. Each step uses only a strictly smaller positive reserve, and finitely many steps reach any prescribed cutoff. Dropping the reserve gives the assertion. □

Explanation of Theorem 4.39. The binomial limit for addition shows that $|f^\sharp(x)|$ defines a valuation on $R^\flat$, so the map $\tau\colon X \to X^\flat$ is well-defined. It is continuous because

$$\tau^{-1}U\left(\frac{f_1,\dots,f_n}{g}\right) = U\left(\frac{f_1^\sharp,\dots,f_n^\sharp}{g^\sharp}\right),$$

using a power of $\varpi^\flat$ among the numerators to preserve the unit-ideal condition.

To go further, the preceding approximation extends to every perfectoid algebra. Indeed, repeated lifting modulo ϖ writes an integral f, to any prescribed precision, as

$$f = h_0^\sharp + \varpi h_1^\sharp + \cdots + \varpi^m h_m^\sharp + \varpi^{m+1}u, \qquad h_i \in R^{\flat\circ}, \quad u \in R^\circ.$$

Apply Proposition 4.41 to the degree-one polynomial $T_0 + \varpi T_1 + \cdots + \varpi^m T_m$, and then send T_i^{1/p^j} to $(h_i^{1/p^j})^\sharp$. Choosing m large absorbs the final error. The same bounds hold for a general R^+: they can be checked at rank-one generizations, and a slight decrease of ϵ supplies a strict margin for their specializations. This gives the general approximation statement of [**Sch12**, Corollary 6.7(i)].

Now consider any rational subset of X, with integral numerators including ϖ^N. On it the denominator has value at least $|\varpi|^N$. Approximate the other numerators and the denominator with cutoff strictly below this bound and with $0 < \epsilon < 1$. Above the cutoff their absolute values are unchanged, and below it they remain smaller than the denominator. Thus this rational subset is the inverse image of a rational subset of $X^\flat$. Since rational subsets distinguish points of the T_0 space X, this proves injectivity of τ.

For surjectivity, let $y \in X^\flat$. Its completed residue field $F = \widehat{k(y)}$ is perfectoid: in characteristic p, the rational localizations just constructed are perfect, and taking the residue field and completing preserves perfectness. The continuous valuation still contains the nondiscrete valuation of $K^\flat$. Untilting $R^\flat \to F$ gives a map from R to a perfectoid field over K; the field construction also transports the valuation ring defining y. This gives $x \in X$ with $\tau(x) = y$. Here one uses that untilting preserves fields and

their valuation rings, as in [Sch12, Proposition 3.6 and Lemma 5.21]. We have proved bijectivity, continuity, and that the rational basis is carried in both directions. Hence τ is a homeomorphism, proving (i).

Every rational subset is now of the form handled by Proposition 4.40. Its universal property, together with Lemma 4.38, identifies the integral subrings as well as the algebras. This proves (ii).

For (iii)–(iv), we invoke the almost Tate acyclicity theorem, precisely [Sch12, Theorem 6.3(iii)–(iv) and Proposition 6.14]. Its proof begins with completed perfections of affinoid algebras in characteristic p, uses Frobenius to make the integral cohomology annihilated by arbitrarily small powers of ϖ, and then passes to general algebras and to mixed characteristic. This analytic step is the external input here; the preceding topological argument alone would not prove it. □

Thus $(\mathcal{O}_X, \mathcal{O}_X^+)$ makes $\mathrm{Spa}(R, R^+)$ an adic space. We call it an *affinoid perfectoid space*.

Definition 4.42. A perfectoid space over K is an adic space over $\mathrm{Spa}(K, K^\circ)$ that is locally affinoid perfectoid. Morphisms of perfectoid spaces are morphisms of adic spaces.

Definition 4.43. A perfectoid space $X^\flat$ over $K^\flat$ is the tilt of X if there are functorial bijections

$$\mathrm{Hom}(\mathrm{Spa}(R^\flat, R^{\flat+}), X^\flat) \cong \mathrm{Hom}(\mathrm{Spa}(R, R^+), X)$$

for every perfectoid affinoid K-algebra (R, R^+).

Theorem 4.44. *Every perfectoid space X over K admits a tilt, unique up to unique isomorphism. Tilting is an equivalence between perfectoid spaces over K and over $K^\flat$, and $|X| \cong |X^\flat|$. A perfectoid space is affinoid if and only if its tilt is affinoid. For every affinoid perfectoid open $U \subset X$, its ring of sections tilts to the ring of sections of $U^\flat$.*

Proof. Choose an affinoid perfectoid open cover of X. Cover its pairwise intersections by rational subsets of the affinoids. Theorem 4.39(ii) makes tilting compatible with these restrictions, so the tilted charts have compatible identifications on their overlaps. Full faithfulness for affinoid pairs ensures that the identifications satisfy the cocycle condition. They therefore glue to $X^\flat$, with the same underlying topological space. Maps glue in the same way. Repeating this construction with the inverse algebra equivalence recovers X, so it is an equivalence of categories. The functorial Hom description gives uniqueness. Applying the two inverse constructions to an affinoid chart proves both the affinoid criterion and the assertion about its sections. □

Fiber products of perfectoid spaces also exist. On affinoid charts they are constructed by completing the tensor product and the corresponding integral subring, and then glued. The fact that the resulting algebra remains perfectoid is the content of [Sch12, Proposition 6.18]; thus this construction gives a fiber product in adic spaces as well as in perfectoid spaces.

6. The Étale Topology

Tilting has given us a homeomorphism of spaces and a correspondence between their rational subsets. The almost purity theorem will allow us to extend this correspondence to their étale sites. First let us recall what it means for a map of adic spaces to be étale.

6.1. Étale maps.

DEFINITION 4.45. A morphism $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is *finite étale* if B is a finite étale A-algebra and B^+ is the integral closure of A^+ in B.

DEFINITION 4.46. A morphism $f : Y \to X$ of adic spaces is finite étale if X has an affinoid open cover $X_i = \mathrm{Spa}(A_i, A_i^+)$ such that each $f^{-1}(X_i)$ is affinoid and its map to X_i is finite étale in the preceding sense.

We will use finite étale algebras with their natural finite-module topology. Over a strongly noetherian affinoid algebra, and, by almost purity below, over a perfectoid algebra, these algebras are sheafy and therefore give adic spaces.

DEFINITION 4.47. A morphism $f : Y \to X$ is *étale* if, for every $y \in Y$, there are open neighborhoods $U \ni y$ and $V \ni f(y)$ such that $f(U) \subset V$ and $U \to V$ factors as an open immersion $U \hookrightarrow W$ followed by a finite étale morphism $W \to V$.

Thus the factorization is local on both source and target. We will verify the required base-change and composition properties in the perfectoid setting. First, it is helpful to keep track of the integral almost rings.

DEFINITION 4.48.
- A morphism $(R, R^+) \to (S, S^+)$ of perfectoid affinoid K-algebras is *strongly finite étale* if it is finite étale and $S^{\circ a}$ is finite étale over $R^{\circ a}$.
- A morphism $f : X \to Y$ of perfectoid spaces is strongly finite étale if Y has an affinoid perfectoid open cover whose inverse images are affinoid perfectoid and whose associated algebra maps are strongly finite étale.
- It is *strongly étale* if locally on source and target it factors as an open immersion followed by a strongly finite étale morphism.

We need two algebraic inputs before using these definitions. First, in characteristic p, a finite étale algebra over a perfectoid algebra is perfectoid, and its integral almost algebra is uniformly finite projective and finite étale; this is [**Sch12**, Proposition 5.23]. The perfectness comes from relative Frobenius for étale algebras, while the almost integral assertion requires the trace-pairing argument. Second, for arbitrary K, the integral tilting equivalence and Theorem 4.23 give a fully faithful functor

$$(R^\flat)_{\text{fét}} \longrightarrow R_{\text{fét}}.$$

Its essential image consists exactly of the finite étale S/R for which S is perfectoid and $S^{\circ a}$ is finite étale over $R^{\circ a}$. Such $S^{\circ a}$ is uniformly finite projective. This is the integral comparison of [**Sch12**, Theorem 5.25], which uses the characteristic-p result, not the general almost purity theorem we are about to prove.

It follows that strongly finite étale and strongly étale morphisms are preserved and reflected by tilting. In characteristic p, they coincide with finite étale and étale morphisms. We are not yet making that assertion in mixed characteristic; establishing it is the point of almost purity.

LEMMA 4.49. (i) *Let $f : X \to Y$ be a strongly finite étale, respectively strongly étale, morphism of perfectoid spaces, and let $Z \to Y$ be a morphism of perfectoid spaces. Then $X \times_Y Z \to Z$ has the same property, and*

$$|X \times_Y Z| \longrightarrow |X| \times_{|Y|} |Z|$$

is surjective.

(ii) *Suppose K has characteristic p, $X \to Y$ is a finite étale, respectively étale, morphism of adic spaces over K, and $Z \to Y$ is a morphism from a perfectoid space. Then the fiber product exists as an adic space, is perfectoid, and its map to Z is finite étale, respectively étale. The same map to the set-theoretic fiber product of underlying spaces is surjective.*

In the affinoid finite case, with $X = \mathrm{Spa}(A, A^+)$, $Y = \mathrm{Spa}(B, B^+)$ and $Z = \mathrm{Spa}(C, C^+)$, the fiber product is

$$\mathrm{Spa}(D, D^+), \qquad D = A \otimes_B C,$$

where D^+ is the integral closure of C^+ in D.

Proof. In the affinoid case, D is finite projective over the complete ring C, so its finite-module topology is already complete: it is a direct summand of some C^n. Tensor products and integral closure give the universal property for maps of Huber pairs. In (i), almost finite étale base change identifies its integral model with

$$A^{\circ a} \otimes_{B^{\circ a}} C^{\circ a}.$$

The integral comparison just stated identifies this model with $D^{\circ a}$ and shows that D is perfectoid. In (ii), the characteristic-p finite étale theorem gives perfectoidness directly. This proves the affinoid assertions.

For a pair of points $x \in X$ and $z \in Z$ over the same point y, the finite étale case reduces to extending the valuations on their residue fields to a factor of

$$k(x) \otimes_{k(y)} k(z).$$

This tensor product is nonzero, and the valuation-extension theorem gives a common valued extension compatible with the prescribed points. One may complete it without changing those restrictions. The resulting valuation is bounded on D^+ and defines a point of the fiber product above (x, z). This proves the asserted surjectivity. It is surjectivity onto pairs with the same image, not surjectivity onto all of Z when $X \to Y$ itself is not surjective.

Finally, the finite statements glue over affinoid charts. For an étale morphism, use its local factorization: open immersions are preserved by base change, and the finite part has just been handled. The same argument with neighborhoods proves the topological assertion. These constructions agree with [**Sch12**, Lemma 7.3]. □

The next two proofs involve a different use of descent. We will write a large algebra as a limit of smaller ones and show that a finite étale cover is already defined at one stage. There are two operations here: taking the filtered colimit, and then completing it. Finite presentation handles the first operation. The second uses a theorem about henselian rings, so we will state that input separately.

If (A_i) is a filtered system of complete flat K°-algebras and A is the completion of $\varinjlim_i A_i$, then

$$(A[1/\varpi])_{\text{fét}} \cong 2\text{-}\varinjlim_i (A_i[1/\varpi])_{\text{fét}}.$$

Here the notation means that objects, morphisms, and equalities between morphisms descend to a sufficiently large finite stage. Before completion, finite presentation reduces this to finitely many generators, relations, and identities: their coefficients occur at one finite stage, and an identity that holds in a filtered colimit holds after increasing that stage. Morphisms and their inverse identities are handled in the same way; see [**The26**, Tag 00QO].

The algebraic limit argument alone does not handle the completion. For this we use the henselian completion theorem [GR03, Proposition 5.4.53]: inverting ϖ in a flat ring henselian along (ϖ) gives the same finite étale category as doing so in its completion. The direct limit of the A_i is henselian along (ϖ), which proves the displayed assertion; this is [Sch12, Lemma 7.5(i)].

In characteristic p, this also shows that a reduced affinoid algebra of topologically finite type and its completed perfection have the same finite étale category. The algebraic reason is that relative Frobenius is an isomorphism for an étale algebra [The26, Tag 0EBS]. Iterating it and applying finite presentation shows that adjoining all p-power roots leaves finite étale covers unchanged. The henselian completion theorem then handles the completion of this filtered union. Finally, an arbitrary perfectoid affinoid algebra in characteristic p can be approximated by such completed perfections, together with any prescribed finite collection of rational domains; we use the precise approximation theorem [Sch12, Lemma 6.13] here.

Proposition 4.50. *Let $f : X \to Y$ be strongly finite étale. For every affinoid perfectoid open $V \subset Y$, the inverse image $U = f^{-1}(V)$ is affinoid perfectoid, and the map*

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \longrightarrow (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is strongly finite étale.

Proof. We may replace Y by V. Tilting preserves strong finite étaleness and affinoidness, so we may also assume that K has characteristic p. Write $Y = \mathrm{Spa}(R, R^+)$. By definition, a finite rational cover of Y trivializes the assertion that the inverse images are affinoid finite étale. The finite étale algebras on these rational subsets, the isomorphisms on their pairwise intersections, and the cocycle identities on triple intersections are finite presentation data.

We need all three parts of this assertion. Descending the algebras alone would give local covers at a finite stage, but would not yet tell us how to glue them. We therefore descend the transition isomorphisms and their inverses, and then enlarge the stage once more so that the cocycle identities hold there. There are only finitely many pieces in our cover, so one stage works for all of them. The identities are finite collections of equalities between finitely presented maps, so the same finite-presentation argument applies. Here we combine it with the analytic approximation of the rational domains.

Apply the approximation and descent results just stated. After passing to one sufficiently large stage, all these data descend to a completed perfection of an affinoid algebra of topologically finite type. Descent through perfection then puts them over the underlying noetherian affinoid adic space Y_0. Thus they glue there to a finite étale adic space $X_0 \to Y_0$. The noetherian affinoid theorem says that X_0 is affinoid; see [Hub96, Example 1.6.6(ii)]. Its pullback to Y is affinoid perfectoid by Lemma 4.49, and it agrees with X on the original rational cover and its gluing maps. It is therefore X.

The characteristic-p integral theorem makes its algebra map strongly finite étale. Untilting proves the assertion over the original field. The use of finite presentation to descend the gluing data is essential here: affinoidness of a finite cover is not a consequence of the definition alone. This is the descent argument underlying [Sch12, Proposition 7.6]. □

PROPOSITION 4.51. *Assume K has characteristic p and let $f\colon X \to Y$ be étale between perfectoid spaces. For each $x \in X$, there are affinoid perfectoid neighborhoods $x \in U \subset X$ and $f(U) \subset V \subset Y$, and an étale map of affinoid noetherian adic spaces $U^0 \to V^0$, such that*

$$U \cong U^0 \times_{V^0} V.$$

PROOF. Choose a local factorization of f as an open immersion into a finite étale space over an affinoid perfectoid neighborhood V. The preceding proposition makes that finite space affinoid. Shrink the source around x to a rational subset of it. We now have a finite étale algebra over $\mathcal{O}_Y(V)$ and finitely many elements defining one rational domain.

Approximate V by completed perfections of affinoid algebras of topologically finite type. Finite étale descent puts the algebra over one stage. The rational-domain data descend after enlarging the stage: sufficiently close approximations to its finitely many defining elements give the same inequalities, because their ideal is open and the denominator is bounded away from zero on the domain. Descent through perfection then gives a finite étale algebra over an ordinary affinoid algebra of topologically finite type and a rational subdomain of its spectrum. Call the base V^0 and the subdomain U^0. Its map is an open immersion followed by a finite étale morphism, hence is étale, and the constructions recover U after base change. This proves the assertion of [**Sch12**, Proposition 7.7]. □

COROLLARY 4.52. *Strongly étale maps of perfectoid spaces are open. Strongly finite étale and strongly étale morphisms are stable under composition.*

PROOF. Tilt to characteristic p. By Proposition 4.51, the map is locally a pullback of a noetherian étale map, which is universally open. The fiber-product description and the surjectivity onto pairs in Lemma 4.49 therefore show that its image on each open neighborhood is open. For composition, descend the finitely many algebra maps and rational-domain data in two local factorizations to a common noetherian stage. There, étale maps compose, and the same is true after pullback. In the finite case, Proposition 4.50 reduces directly to the transitivity of finite étale ring maps. Tilting back proves all the assertions. The noetherian inputs are [**Hub96**, Propositions 1.7.8 and 1.6.7(ii)]. □

We can now pass from the characteristic-p and field cases to almost purity for an arbitrary perfectoid algebra.

THEOREM 4.53 (Almost purity). *Let (R, R^+) be perfectoid and $X = \mathrm{Spa}(R, R^+)$. Then:*

(i) *For every affinoid perfectoid open $U \subset X$, taking global sections gives a fully faithful functor from the* opposite *of the category of strongly finite étale spaces over U to the category of finite étale $\mathcal{O}_X(U)$-algebras.*
(ii) *This functor is an equivalence. Thus global sections give a contravariant equivalence between those spaces and finite étale algebras.*
(iii) *Every finite étale R-algebra S, with its natural topology, is perfectoid; $S^{\circ a}$ is finite étale and uniformly almost finitely generated over $R^{\circ a}$.*

PROOF. We make explicit the local comparison used in this argument. For a point $x \in X$, finite étale descent and henselian completion give

$$2\text{-}\varinjlim_{x \in U} (\mathcal{O}_X(U))_{\text{fét}} \cong \widehat{k(x)}_{\text{fét}},$$

where U runs through affinoid neighborhoods. Finite presentation descends an algebra and its maps from the local ring to a neighborhood; the henselian completion theorem identifies the resulting finite étale category with that of the completed residue field. This is the application of [**Sch12**, Lemma 7.5(i)] used in its Theorem 7.9. The completed residue field is perfectoid by [**Sch12**, Corollary 6.7(ii)]. These are the local comparison inputs, in addition to the field tilting theorem stated at the start of this chapter.

For (i), Proposition 4.50 makes every object affinoid over U. The integral comparison identifies its algebra with a finite étale $\mathcal{O}_X(U)^{\circ a}$-algebra, and inverting ϖ is fully faithful on this category. Affinoid maps reverse the direction of ring maps, which explains the opposite category in the statement.

For (ii), replace U by X and fix a finite étale R-algebra S. At a point x, combine the local comparison with field tilting to obtain

$$2\text{-}\varinjlim_{x\in U}(\mathcal{O}_X(U))_{\text{fét}} \cong 2\text{-}\varinjlim_{x^\flat\in U^\flat}(\mathcal{O}_{X^\flat}(U^\flat))_{\text{fét}}.$$

Thus S comes, on a sufficiently small neighborhood of x, from a finite étale algebra over the tilt. In characteristic p that algebra is strongly finite étale. Untilting gives a strongly finite étale space over U, whose algebra is isomorphic to $S\otimes_R\mathcal{O}_X(U)$ after shrinking U again if necessary. The shrinking is justified by the descent of morphisms and identities as well as objects.

Choose finitely many such rational neighborhoods U_i covering X, and write $S_i = S\otimes_R\mathcal{O}_X(U_i)$. Put S_i^+ equal to the integral closure of $\mathcal{O}_X^+(U_i)$ in S_i. On overlaps, full faithfulness in (i) turns the canonical identifications of these algebras into identifications of $\mathrm{Spa}(S_i,S_i^+)$. Their cocycle condition follows from the same full faithfulness. They glue to a strongly finite étale $Y\to X$, and Proposition 4.50 makes Y affinoid perfectoid, say $Y=\mathrm{Spa}(A,A^+)$.

It remains to identify A with S. The sheaf condition on Y gives

$$0\longrightarrow A\longrightarrow\prod_i S_i\longrightarrow\prod_{i,j}S\otimes_R\mathcal{O}_X(U_i\cap U_j).$$

The corresponding equalizer for $\mathcal{O}_X$ is

$$0\longrightarrow R\longrightarrow\prod_i\mathcal{O}_X(U_i)\longrightarrow\prod_{i,j}\mathcal{O}_X(U_i\cap U_j).$$

Since the cover is finite and S is a flat R-module, tensoring the second sequence with S preserves its exactness and its finite products. Comparing the two kernels gives $A\cong S$. The construction and (i) prove the desired anti-equivalence.

Finally, (ii) realizes S as the algebra of a strongly finite étale affinoid perfectoid space. This proves perfectoidness and the almost finite étale assertion in (iii). The uniform bound is the uniform finite projectivity assertion in the integral comparison, [**Sch12**, Theorem 5.25]; it is not merely a consequence of ordinary finite generation after inverting ϖ. This completes the argument of [**Sch12**, Theorem 7.9]. □

We now know in either characteristic that the (finite) étale and strongly (finite) étale notions agree. In particular, the proof of Lemma 4.49(ii) shows that any étale adic space over a perfectoid space is perfectoid.

DEFINITION 4.54. For a perfectoid space X, its étale site $X_{\text{ét}}$ consists of spaces étale over X; by the preceding result these are perfectoid. Covering families are jointly surjective families of morphisms over X.

THEOREM 4.55. *Tilting induces an equivalence of sites $X_{\acute{e}t} \cong X^{\flat}_{\acute{e}t}$, functorial in X.*

PROOF. Tilting and untilting preserve open immersions and strongly finite étale morphisms. By almost purity and the local factorization of étale maps, they therefore give inverse equivalences on the étale objects and their morphisms. The natural homeomorphisms of the underlying spaces also preserve joint surjectivity in both directions, so the equivalence preserves coverings. □

CHAPTER 5

The Hodge-Tate Decomposition

We are finally ready to use the perfectoid spaces of the last chapter to study the cohomology of a rigid space. The idea is to work with a topology in which we can cover our space by objects that, after completion, are perfectoid. This will let us make sheaf-theoretic versions of the period rings of Chapter 2 and use them in the comparison. We mainly follow [**Sch13b**], with the covering conventions and construction of the structural de Rham sheaf taken from the corrigendum [**Sch16**].

1. The Pro-Étale Site

We saw in the last chapter that adjoining infinitely many roots can turn an affinoid into a perfectoid space. To keep track of the entire tower of covers over the original space, it is convenient to work with formal inverse limits. For this, we recall the notion of a pro-category.

Definition 5.1. Let $\mathcal{C}$ be any category. The pro-category $\mathrm{Pro}(\mathcal{C})$ is the category whose objects are functors $F\colon I \to \mathcal{C}$, where I is a small, cofiltered index category, and for any two $F\colon I \to \mathcal{C}$ and $G\colon J \to \mathcal{C}$, we set

$$\mathrm{Hom}(F,G) = \varprojlim_J \varinjlim_I \mathrm{Hom}(F(i),G(j))$$

Let us apply this to a locally noetherian adic space X, that is, a space locally of the form $\mathrm{Spa}(A,A^+)$ with A strongly noetherian. One useful fact in this setting is that an adic space étale over X is locally connected. We will use this repeatedly in studying towers of finite étale covers.

Before defining the full pro-étale site, we look at the slightly simpler situation of towers of finite étale covers, where we have a description in terms of the fundamental group. Write $X_{\text{fét}}$ for the category of adic spaces finite étale over X. For an object $U = \varprojlim U_i \in \mathrm{Pro}(X_{\text{fét}})$, we have the associated topological space $|U| = \varprojlim |U_i|$.

Definition 5.2. Call a morphism $V \to U$ in $\mathrm{Pro}(X_{\text{fét}})$ an admissible tower if

$$V = \varprojlim_{\mu<\lambda} V_\mu, \qquad V_0 = U,$$

for an ordinal λ, and, for every $0 < \mu < \lambda$, the map

$$V_\mu \longrightarrow V_{<\mu} := \varprojlim_{\mu'<\mu} V_{\mu'}$$

is the pullback of a finite étale surjection at a finite level. The pro-finite étale site $X_{\text{profét}}$ is $\mathrm{Pro}(X_{\text{fét}})$ with the topology generated by finite jointly surjective families of finite étale maps and by these admissible towers. In particular, at a limit ordinal we impose a condition relative to the whole preceding limit, not merely relative to each earlier term.

Suppose now that X is connected, and fix a geometric basepoint $\bar{x}$ of X. Recall that finite étale covers of X are equivalent to finite sets with continuous $\pi_1(X, \bar{x})$-action. Since we are now looking at inverse limits of these covers, we should expect profinite sets to enter the picture.

Definition 5.3. Let G be a profinite group. We let $\mathrm{FinSet}(G)$ denote the site whose underlying category is the category of finite sets with continuous G-action, and coverings are given by families of G-equivariant maps $\{f_i \colon S_i \to S\}$ such that $S = \cup_i f_i(S_i)$. We let $\mathrm{ProFinSet}(G)$ denote the category of profinite sets with continuous G-action, with the analogous topology: it is generated by finite jointly surjective families pulled back from finite G-sets and by ordinal-indexed towers whose maps to the preceding limit are pullbacks of surjections of finite G-sets.

Proposition 5.4. *Let X be a connected locally noetherian adic space. Then there is a canonical equivalence of sites*

$$X_{\mathit{prof\acute{e}t}} \cong \mathrm{ProFinSet}(\pi_1(X, \bar{x}))$$

Proof. Put $G = \pi_1(X, \bar{x})$. The fibre functor identifies $X_{\text{fét}}$ with $\mathrm{FinSet}(G)$. Passing to pro-categories identifies their objects and morphisms, by the formula defining Hom in a pro-category. Every profinite G-set is the inverse limit of its finite continuous G-equivariant quotients, so $\mathrm{Pro}(\mathrm{FinSet}(G)) = \mathrm{ProFinSet}(G)$. Finally, the fibre functor preserves finite limits and finite étale surjections. It therefore identifies both the finite covering families and every stage of an admissible tower. The equivalence is consequently an equivalence of sites, with the covering convention just specified. □

Throughout this chapter, a local system with finite coefficient ring is understood to have finite rank (or, for torsion modules, a finite stalk).

There is a nice way to interpret continuous group cohomology using this site. For a topological G-module M, define a sheaf $\mathcal{F}_M$ on $\mathrm{ProFinSet}(G)$ by $\mathcal{F}_M(S) = \mathrm{Hom}_{\mathrm{cont},G}(S, M)$. Taking sections over a point is just taking G-invariants. As one might expect, taking higher cohomology will give us continuous group cohomology.

Proposition 5.5. *With notation as above, we have:*

(i) *An admissible tower $S \to S'$ of profinite sets admits a continuous splitting. The splitting need not respect a group action.*
(ii) *For any $S \in \mathrm{ProFinSet}(G)$ with free G-action, the functor $\mathcal{F} \mapsto \mathcal{F}(S)$ on sheaves over $\mathrm{ProFinSet}(G)$ is exact.*
(iii) *There is a canonical isomorphism $H^i(\mathrm{pt}, \mathcal{F}_M) \cong H^i_{\mathrm{cont}}(G, M)$ for all $i \geqslant 0$. Here pt is the one-point set with trivial G-action.*

Proof. For (i), choose a section at each stage by transfinite induction. A surjection of finite sets has a section, and its pullback therefore has a section. At a limit stage the previously chosen compatible sections give a map into the inverse limit; composing this with a section of the next finite pullback continues the construction. Taking the final inverse limit gives a continuous section $S' \to S$. This argument explains why the condition at limit ordinals is necessary; the assertion for all open surjections would be false.

For (ii), first observe that a free profinite G-set is of the form $T \times G$, noncanonically. Indeed, write G as a transfinite tower of its finite quotients. The corresponding quotients

of the free G-set express $S \to S/G$ as an admissible tower: at each step a finite group acts freely, and a finite covering of a profinite set is trivial over a finite clopen partition. Part (i) gives a section of $S \to S/G$, and hence the claimed description. Now let $\mathcal{F} \to \mathcal{F}'$ be an epimorphism, and let $s' \in \mathcal{F}'(T \times G)$. Locally, s' has lifts. Pull the covering family back along $T \to T \times G$. Finite covering families can be split after a finite clopen partition of T, and admissible towers split by (i). Pulling the local lifts back along these sections and then translating by G gives an equivariant splitting of a refining cover. Thus s' lifts globally. Left exactness of sections is automatic, proving exactness.

For (iii), the map $G \to \mathrm{pt}$ is an allowed cover: adjoining the finite quotients of G one at a time gives the required admissible tower. Its iterated fibre products are G^{q+1} with diagonal action, which is free. By (ii), they are acyclic for every abelian sheaf. The Čech complex therefore computes $H^*(\mathrm{pt}, \mathcal{F}_M)$, and its degree q term is

$$\mathrm{Hom}_{\mathrm{cont},G}(G^{q+1}, M) \cong \mathrm{Hom}_{\mathrm{cont}}(G^q, M).$$

The alternating face maps give the usual differential computing continuous group cohomology. This gives the asserted isomorphism, including its functoriality in M. The same complex will appear when we use Galois covers below. □

In doing geometry we would also like to be able to work over small étale neighborhoods, something that the finite étale covers alone do not give us. We therefore allow a little more than in the pro-finite étale site and arrive at the pro-étale site $X_{\mathrm{proét}}$.

Definition 5.6. A morphism $U \to V$ of objects in $\mathrm{Pro}(X_{\text{ét}})$ is called (finite) étale if it is induced by a (finite) étale morphism $U_0 \to V_0$ of objects in $X_{\text{ét}}$ such that $U = U_0 \times_{V_0} V$ via some morphism $V \to V_0$. A morphism $U \to V$ is called pro-étale if it can be written as a cofiltered inverse limit $U = \varprojlim U_i$ of objects $U_i \to V$ étale over V, such that $U_i \to U_j$ is finite étale and surjective for sufficiently large $i > j$.

The pro-étale site $X_{\mathrm{proét}}$ has as its underlying category the full subcategory of $\mathrm{Pro}(X_{\text{ét}})$ consisting of objects that are pro-étale over X. Its topology is generated by jointly surjective étale covering families and by admissible towers of finite étale surjections, with the same condition at limit ordinals as above. A covering may be refined by composites of coverings of these two types. This is the corrected topology of [**Sch16**]; we have not changed the underlying category of pro-étale objects. All the countable towers used below are coverings in this topology.

Lemma 5.7. *We verify that the pro-étale site is indeed a site.*

(i) *Let $U, V, W \in \mathrm{Pro}(X_{ét})$, and assume that $U \to V$ is (finite, pro-) étale. Let $W \to V$ be any morphism. Then the fiber product $U \times_V W$ exists in $\mathrm{Pro}(X_{ét})$, and the canonical map $U \times_V W \to W$ is (finite, pro-) étale. Moreover, the map $|U \times_V W| \to |U| \times_{|V|} |W|$ on underlying spaces is surjective.*

(ii) *Any composition $U \to V \to W$ of (finite) étale morphisms in $\mathrm{Pro}(X_{ét})$ is again (finite) étale.*

(iii) *Let $U \in \mathrm{Pro}(X_{ét})$ be any object and $W \subset |U|$ be a quasicompact open subset. Then there is some other $V \in \mathrm{Pro}(X_{ét})$, together with an étale map $V \to U$ such that $|V| \to |U|$ induces a homeomorphism $|V| \cong W$.*

(iv) *Any pro-étale map $U \to V$ in $\mathrm{Pro}(X_{ét})$ is open.*

(v) *A surjective étale (resp. surjective finite étale) map $U \to V$ in $\mathrm{Pro}(X_{\acute{e}t})$ with $V \in X_{pro\acute{e}t}$ arises via pullback along $V \to V_0$ from a surjective étale (resp. surjective finite étale) map $U_0 \to V_0$ of objects U_0, V_0 in $X_{\acute{e}t}$.*

(vi) *Let $U \to V \to W$ be pro-étale morphisms in $\mathrm{Pro}(X_{\acute{e}t})$, and assume $W \in X_{pro\acute{e}t}$. Then $U, V \in X_{pro\acute{e}t}$ and the composition $U \to W$ is pro-étale.*

(vii) *All finite projective limits exist in $X_{pro\acute{e}t}$.*

Proof. We proceed in order, keeping track of the distinction between fibre products of spaces and fibre products of their underlying sets.

(i) In the étale case, descend the map to $U_0 \to V_0$ and write the base change as $\varprojlim_j (U_0 \times_{V_0} W_j)$. This represents the fibre product in the pro-category. There is a natural map

$$\varprojlim_j |U_0 \times_{V_0} W_j| \longrightarrow \varprojlim_j \big(|U_0| \times_{|V_0|} |W_j|\big) = |U| \times_{|V|} |W|.$$

The arrow need not be an equality. Given a point on the right, the choices of a point above it at a finite stage form nonempty finite sets; if necessary we first restrict to a quasicompact étale neighbourhood of the specified point. These finite sets form an inverse system, whose limit is nonempty by compactness. Thus the arrow is surjective. For a pro-étale map, take the additional inverse limit over its étale presentation. The fibres are now inverse limits of nonempty compact sets, so the same argument gives surjectivity. Base change preserves finite étale maps and surjectivity, giving the remaining assertions.

(ii) Descend both maps to finite levels. The finitely many maps and relations needed to describe the second pullback already occur at a common sufficiently large level. There their composite is (finite) étale, and pulling back this composite gives $U \to W$.

(iii) Quasicompact opens in the inverse limit topology descend to an open $W_i \subset |U_i|$ at some finite level. Set $V = W_i \times_{U_i} U$, viewing W_i as an open adic subspace. Open immersions commute with base change and induce the asserted homeomorphism. If U is pro-étale, this construction gives a pro-étale presentation of V as well.

(iv) A quasicompact open in a pro-étale presentation descends to a finite stage. After passing beyond the stage at which the transition maps are surjective, its image is the image of that finite-stage open. An étale map of adic spaces is open. For its base change in the pro-category, part (i) identifies its image with the inverse image of that open image. Thus every pro-étale map is open.

(v) Write $U = U_0 \times_{V_0} V$ and $V = \varprojlim V_i$. The open image of $U_0 \to V_0$ contains the image of $V \to V_0$, by (i). The projections $V \to V_i$ are surjective for sufficiently large i, so the map $V_i \to V_0$ has image in this same open subset. Consequently $U_0 \times_{V_0} V_i \to V_i$ is a surjective (finite) étale map representing $U \to V$.

(vi) Factor a pro-étale presentation into an étale map followed by an inverse limit of finite étale surjections. Composition with the étale part follows from (i) and (ii), after descending it to a finite stage.

For the remaining assertion we may work over a connected étale neighbourhood and use its fundamental group G. We then have open surjections of profinite G-sets. Their composite is again open and surjective. To express such

a map $S \to T$ as a pro-étale presentation, start with a finite clopen partition of S. Its images are clopen in T, since the map is open and the sets are compact. Refine T by a finite clopen partition subordinate to these images, and refine S by its inverse image. Refining further by finitely many G-translates makes the partitions G-invariant. We obtain finite G-sets $A \to B$ and a surjection $S \to A \times_B T$. These refinements form a cofiltered system with limit S. They therefore express $S \to T$ as an inverse limit of finite étale surjections. This proves stability under composition and, applying the same argument to $W \to X$, shows that $U, V \in X_{\text{proét}}$. This statement concerns morphisms of the underlying category; admissibility of covering towers is checked separately by concatenating their ordinal presentations.

(vii) The terminal object is X, and finite products follow from (i) and (vi). To construct an equalizer, first localize at the initial étale stages so that both maps have image in a common separated étale neighbourhood. The remaining finite étale stages have open and closed diagonals; hence, on this neighbourhood, the desired equalizer is an intersection of open and closed subobjects in a pro-étale presentation. These local equalizers glue. At a finite stage its image is a union of connected components. Locally noetherian étale objects have locally only finitely many connected components, so this image is open and closed. The transition maps between these images are finite étale and surjective after discarding initial stages. Their inverse limit is the desired equalizer and is pro-étale over X.

Finally, ordinary étale coverings are stable under base change and composition. The same is true of admissible towers, by pulling back their stages and by concatenating the towers. Thus the covering convention does give a Grothendieck topology. □

LEMMA 5.8. *Under the fully faithful embedding of categories $X_{\text{profét}} \subset X_{\text{proét}}$, a morphism $f : U \to V$ in $X_{\text{profét}}$ is open if and only if it is pro-étale as a morphism in $X_{\text{proét}}$. With the admissible-tower convention on both sides, the induced covering topologies coincide, so there is a natural map of sites $X_{\text{proét}} \to X_{\text{profét}}$. Openness alone is not the covering criterion for unrestricted inverse limits.*

PROOF. A pro-étale map is open by the preceding lemma. Conversely, over each connected component of X, translate the map into profinite sets with fundamental-group action. The finite-clopen-partition construction in part (vi) of that lemma expresses an open map, over its clopen image, as a cofiltered limit of finite étale maps with surjective transition maps. The inclusion of the image is étale, giving a pro-étale presentation of the original map. Finally, finite étale covering families and admissible towers are preserved by the inclusion, and their pullbacks remain of the same kind. These generators give the asserted agreement of topologies. □

The comparison theorem concerns geometric cohomology, so we will frequently need to extend the ground field, sometimes by an infinite algebraic extension. Suppose that X has a structure map $X \to \mathrm{Spa}(K, K^+)$. Let L/K be an algebraic separable extension in which $L^+ \subset L$ is the integral closure of K^+. Write X_L for the analytic base change to the completion of (L, L^+), when considering its analytic or étale site. We also define an object of $X_{\text{proét}}$ by taking the inverse limit of $X_{L_i} \in X_{\text{ét}}$ where $L_i \subset L$ run through the finite extensions of K. We let (by abuse of notation) X_L denote this formal inverse limit.

We now have two notions of working over L: we can first extend scalars and then take the pro-étale site, or stay on the original site and look at objects over X_L. It turns out that these amount to the same thing.

PROPOSITION 5.9. *There is an equivalence of sites $X_{L,pro\acute{e}t} \cong X_{pro\acute{e}t}/X_L$.*

PROOF. Write $L = \bigcup_i L_i$ with L_i/K finite. The étale descent theorem for an algebraic extension and its completion says that an étale object, together with finitely many morphisms and relations, has a model over some X_{L_i} after an étale refinement. This is the limit theorem used in [**Sch13b**, Proposition 3.15]; it is not a claim that every analytic function over the completed extension descends to a finite field extension. The same finite-presentation descent applies to finite étale maps and to their surjectivity after increasing i.

Let us use this descent result in the two directions of the comparison. A pro-étale presentation over X_L determines, by taking a common cofiltered system of finite levels, an object of $X_{\mathrm{pro\acute{e}t}}$ equipped with a map to X_L. Conversely, such a map supplies compatible base-change maps to every X_{L_i}, and the same descent construction recovers a presentation over L. The two constructions are inverse on objects and on morphisms by the defining limit–colimit formula for a pro-category. They preserve étale covering families and every finite stage of an admissible tower, so they identify the covering topologies. For completed algebraic extensions the corresponding statement is obtained using the usual étale descent through completion, as in [**Sch13b**, Proposition 3.15]. □

Note that there is a natural morphism of sites $\nu\colon X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ corresponding to the inclusion of categories

$$\begin{array}{ccc} X_{\mathrm{\acute{e}t}} & \hookrightarrow & \mathrm{Pro}(X_{\mathrm{\acute{e}t}}) \\ & \searrow \quad \nearrow & \\ & X_{\mathrm{pro\acute{e}t}} & \end{array}$$

The pro-étale site is much larger than the étale site, but it retains the cohomology of the sheaves we started with on the étale site. The following lemma gives a more general statement: it allows us to compute the cohomology on a pro-object from the cohomology at each stage of its presentation.

LEMMA 5.10. *Let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}t}$. For any quasicompact, quasiseparated object $U = \varprojlim U_j$ in $X_{pro\acute{e}t}$ and $i \geqslant 0$, we have*

$$H^i(U, \nu^*\mathcal{F}) \cong \varinjlim H^i(U_j, \mathcal{F})$$

PROOF. We may assume that $\mathcal{F}$ is injective and X is qcqs. Let $X_{\mathrm{pro\acute{e}tqc}}$ be the subsite of $X_{\mathrm{pro\acute{e}t}}$ consisting of quasicompact objects in $X_{\mathrm{pro\acute{e}t}}$. Then $X_{\mathrm{pro\acute{e}tqc}}$ and $X_{\mathrm{pro\acute{e}t}}$ have the same associated topos, so we may work completely with $X_{\mathrm{pro\acute{e}tqc}}$.

Let $\tilde{\mathcal{F}}$ be the presheaf on $X_{\mathrm{pro\acute{e}tqc}}$ given by $\tilde{\mathcal{F}}(V) = \varinjlim \mathcal{F}(V_j)$, where $V = \varprojlim V_j$. Then $\nu^*\mathcal{F}$ is the sheafification of $\tilde{\mathcal{F}}$, and we must show that $\tilde{\mathcal{F}}$ is a sheaf with $H^i(V, \tilde{\mathcal{F}}) = 0$ for all $V \in X_{\mathrm{pro\acute{e}tqc}}$ and $i > 0$. By SGA 4, Exposé V, Proposition 4.3, it suffices to show that for any $U \in X_{\mathrm{pro\acute{e}tqc}}$ with pro-étale covering $V_k \to U$, the corresponding Čech complex

$$0 \to \tilde{\mathcal{F}}(U) \to \prod_k \tilde{\mathcal{F}}(V_k) \to \prod_{k,k'} \tilde{\mathcal{F}}(V_k \times_U V_{k'}) \to \cdots$$

is exact.

Since U is quasicompact, we may pass to a finite subcover, which may be combined into a single morphism $V \to U$. Let $V = \varprojlim V_l \to U$ be a pro-étale representation of V. For large l, the maps $V_l \to U$ are étale covers. By the definition of $\tilde{\mathcal{F}}$, the Čech complex for $V \to U$ is the direct limit of all of the Čech complexes $V_l \to U$. So we are reduced to the case where $V \to U$ is étale.

Choose $V_j \to U_j$ étale such that $V = V_j \times_{U_j} U$. For $j' \geqslant j$, let $V_{j'} \to U_{j'}$ denote the pullback of $V_j \to U_j$. Then $V_{j'} \to U_{j'}$ is an étale cover for large j', and again the Čech complex for $V \to U$ is the direct limit over j' of the Čech complexes for $V_{j'} \to U_{j'}$. So we are reduced to checking exactness for $V_{j'} \to U_{j'}$, which follows immediately from the acyclicity of the injective sheaf $\mathcal{F}$ on $X_{\text{ét}}$. □

In particular, we can apply the lemma to the étale objects themselves and get a good deal of information about the direct image under ν.

Corollary 5.11. *Let $\mathcal{F}$ be an abelian sheaf on $X_{ét}$.*

(i) *The adjunction map $\mathcal{F} \to R\nu_*\nu^*\mathcal{F}$ is an isomorphism.*
(ii) *For any qcqs morphism $f : X \to Y$, the base change map*

$$\nu_Y^* R f_{ét\,*}\mathcal{F} \to R f_{proét\,*}\nu_X^*\mathcal{F}$$

associated to the diagram

$$\begin{array}{ccc} X_{proét} & \xrightarrow{\nu_X} & X_{ét} \\ \downarrow{\scriptstyle f_{proét}} & & \downarrow{\scriptstyle f_{ét}} \\ Y_{proét} & \xrightarrow{\nu_Y} & Y_{ét} \end{array}$$

is an isomorphism.

Proof. We proceed in order:

(i) For any $i \geqslant 0$, $R^i\nu_*\nu^*\mathcal{F}$ is the sheafification of the presheaf $U \mapsto H^i(U, \nu^*\mathcal{F})$. The above lemma gives the desired isomorphism for $i = 0$. Moreover, the lemma says that if U is qcqs and $i > 0$, there is an isomorphism $H^i(U, \nu^*\mathcal{F}) \cong H^i(U, \mathcal{F})$. But any section of $H^i(U, \mathcal{F})$ vanishes locally on $U_{\text{ét}}$, so the sheafification is 0.
(ii) For $i \geqslant 0$, the i-th cohomology of both sides is the sheafification of the presheaf taking a qcqs $U = \varprojlim U_j \to Y$ to

$$\varinjlim H^i(U_j \times_Y X, \mathcal{F})$$

□

So for a sheaf that came from the étale site, pulling back and taking the derived direct image does nothing to it. In particular, its cohomology does not change.

Corollary 5.12. *ν^* is fully faithful on abelian sheaves, and induces an isomorphism $H^i_{ét}(X, \mathcal{F}) \cong H^i_{proét}(X, \nu^*\mathcal{F})$.*

Proof. For abelian sheaves $\mathcal{F}, \mathcal{G}$ on $X_{\text{ét}}$, adjunction and the preceding corollary give

$$\operatorname{Hom}(\nu^*\mathcal{F}, \nu^*\mathcal{G}) = \operatorname{Hom}(\mathcal{F}, \nu_*\nu^*\mathcal{G}) = \operatorname{Hom}(\mathcal{F}, \mathcal{G}).$$

Applying derived global sections to $\mathcal{F} \simeq R\nu_*\nu^*\mathcal{F}$ gives the cohomology comparison in every degree. □

We will soon construct sheaves by taking inverse limits, for example over the quotients by p^n. The sections of an inverse limit are easy to describe: they are compatible families of sections. Exactness is subtler. A section may lift at every finite level without admitting compatible lifts at all levels.

For a countable system of abelian groups, the obstruction is measured by $\varprojlim^1$. For a system (M_n) with transition maps f_n, this group is the cokernel of $\prod_n M_n \to \prod_n M_n$, $(x_n) \mapsto (x_n - f_n(x_{n+1}))$. If all f_n are surjective, an equation $x_n - f_n(x_{n+1}) = y_n$ can be solved recursively, so this cokernel is zero. The same conclusion holds under the Mittag–Leffler condition, which says that the images in each fixed M_n eventually stabilize; see [**The26**, Tag 07KV].

For a countable system of complexes D_n, the same difference map gives the cohomology-limit exact sequence

$$0 \longrightarrow \varprojlim_n{}^{1} H^{j-1}(D_n) \longrightarrow H^j(\mathbf{R}\varprojlim_n D_n) \longrightarrow \varprojlim_n H^j(D_n) \longrightarrow 0.$$

Indeed, the derived inverse limit is represented by the cone of $1 -$ shift on the product of the complexes, shifted by -1; taking its long exact cohomology sequence gives the assertion. Thus a Mittag–Leffler condition on cohomology removes the extra term when passing to the limit.

The following lemma combines this condition on sections with acyclicity on a basis of the site. It is this combination that will let us pass to completed period sheaves.

Lemma 5.13. *Let $\mathcal{F}_i$ be an $\mathbf{N}$-indexed inverse system of abelian sheaves on a site T. Suppose that there is a basis B for the site T such that for any $U \in B$, the following two conditions hold:*

(i) *The higher limit $R^1 \varprojlim_i \mathcal{F}_i(U)$ vanishes.*
(ii) *All cohomology groups $H^j(U, \mathcal{F}_i)$ vanish for $j > 0$.*

Then $R^j \varprojlim \mathcal{F}_i = 0$ for $j > 0$ and $(\varprojlim \mathcal{F}_i)(U) \cong \varprojlim \mathcal{F}_i(U)$ for all $U \in T$. Finally, $H^j(U, \varprojlim \mathcal{F}_i) = 0$ for $U \in B$ and $j > 0$.

Proof. For an inverse system of sheaves there are two ways to derive the equality

$$\Gamma(U, \varprojlim_i \mathcal{F}_i) = \varprojlim_i \Gamma(U, \mathcal{F}_i).$$

The resulting spectral sequence computes $R\Gamma(U, R\varprojlim_i \mathcal{F}_i)$ from

$$R^p \varprojlim_i H^q(U, \mathcal{F}_i).$$

One way to justify it is to take an injective resolution in the category of inverse systems. The constant-system functor is exact and is left adjoint to inverse limit, so inverse limit preserves injectives; evaluation and section functors give the other composite. For a countable inverse system of abelian groups, $R^p \varprojlim$ vanishes for $p > 1$.

If $U \in B$, hypothesis (ii) removes all rows $q > 0$, and hypothesis (i) removes the remaining term of total degree one. Thus $R\Gamma(U, R\varprojlim_i \mathcal{F}_i)$ is concentrated in degree zero. Sheafifying these cohomology groups as U varies over a basis shows that $R^j \varprojlim_i \mathcal{F}_i = 0$ for $j > 0$. The underived inverse limit of sheaves is calculated sectionwise, giving the

asserted formula for every U. Substituting $R\varprojlim_i \mathcal{F}_i = \varprojlim_i \mathcal{F}_i$ into the same calculation proves its acyclicity on B. □

2. Sheaves on the Pro-Étale Site

We will need several sheaves of topological rings on the pro-étale site. Before constructing the period sheaves themselves, let us look at the structure sheaf. It is especially important here to distinguish it from its completion: on a perfectoid object, it is the latter that will give us the functions on the associated perfectoid space.

Definition 5.14. Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. We define the following sheaves on $X_{\text{proét}}$.

- The (uncompleted) structure sheaf $\mathcal{O}_X = \nu^* \mathcal{O}_{X_{\text{ét}}}$, with subring of integral elements $\mathcal{O}_X^+ = \nu^* \mathcal{O}_{X_{\text{ét}}}^+$.
- The integral completed structure sheaf $\widehat{\mathcal{O}}_X^+ = \varprojlim \mathcal{O}_X^+/p^n$, and the completed structure sheaf $\widehat{\mathcal{O}}_X = \widehat{\mathcal{O}}_X^+[1/p]$.

We can understand the completed structure sheaves rather concretely by looking at the objects of the site that give rise to affinoid perfectoid spaces. For now, work over a perfectoid field K of characteristic zero, with open and bounded valuation ring $K^+ \subset K$, and let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$. Let $\Gamma = |K^\times| \subset \mathbf{R}_{>0}$, and use the logarithm with base $|p|$ to identify $\mathbf{R}_{>0}$ with $\mathbf{R}$. For $r \in \log \Gamma$, fix an element $p^r \in K$ satisfying $|p^r| = |p|^r$.

Definition 5.15. Fix some $U \in X_{\text{proét}}$.

(i) U is *affinoid perfectoid* if U has a pro-étale presentation $U = \varprojlim U_i \to X$ by affinoid $U_i = \mathrm{Spa}(R_i, R_i^+)$ such that (R, R^+) is a perfectoid affinoid (K, K^+)-algebra, where R^+ is the p-adic completion of $\varinjlim R_i^+$, and $R = R^+[1/p]$.

(ii) U is *perfectoid* if it has an open cover by affinoid perfectoid $V \subset U$.

Example 5.16. Let

$$X = \mathbf{T}^n = \mathrm{Spa}(K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, K^+\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$$

Then the inverse limit $\tilde{\mathbf{T}}^n \in X_{\text{proét}}$, defined by

$$\varprojlim_m \mathrm{Spa}(K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle, K^+\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle)$$

is affinoid perfectoid.

In the example above, although the tower is a formal object of the site, we can complete its ring of functions to get an honest perfectoid space. The same construction works for any affinoid perfectoid object U as in (i): we associate to it $\hat{U} = \mathrm{Spa}(R, R^+)$. It is well defined, and gluing gives a functor from the full subcategory of perfectoid objects in $X_{\text{proét}}$ to perfectoid spaces over K which preserves the underlying topological spaces.

Lemma 5.17. *Let $U = \varprojlim U_i \in X_{\text{proét}}$ be affinoid perfectoid, with each $U_i = \mathrm{Spa}(R_i, R_i^+)$ affinoid. Let (R, R^+) be the completion of the direct limit of the R_i, and let $\hat{U} = \mathrm{Spa}(R, R^+)$.*

Let $V_i = \mathrm{Spa}(S_i, S_i^+) \to U_i$ be an étale map such that it can be written as a composition of rational subsets and finite étale maps. For $j \geqslant i$, let $V_j = V_i \times_{U_i} U_j = \mathrm{Spa}(S_j, S_j^+)$, and

$V = V_i \times_{U_i} U = \varprojlim V_j \in X_{proét}$. Finally, let A_j be the p-completion of the p-torsion-free quotient of $S_j^+ \otimes_{R_j^+} R^+$. Then:

(i) *The completion (S, S^+) of the direct limit of the (S_j, S_j^+) is a perfectoid affinoid (K, K^+)-algebra, and hence V is affinoid perfectoid. Moreover, $\hat{V} = V_j \times_{U_j} \hat{U}$, and $S = A_j[p^{-1}]$ for any $j > i$.*

(ii) *For any $j \geqslant i$, the cokernel of the map $A_j \to S^+$ is annihilated by some large power p^N of p.*

(iii) *Fix $\epsilon \in \log \Gamma$ greater than zero. There is some j such that the cokernel of the map $A_j \to S^+$ is annihilated by p^ϵ.*

Proof. We may treat a rational localization and a finite étale map separately, and then compose the conclusions. Write $\hat{U} = \mathrm{Spa}(R, R^+)$ and let $W = V_j \times_{U_j} \hat{U}$. Rational localization preserves perfectoidness, and finite étale extensions do so by almost purity. Thus $W = \mathrm{Spa}(T, T^+)$ is perfectoid. The universal property of the completed tensor product shows that $T = A_j[1/p]$. Both the image of A_j and T^+ are open bounded integral subrings of T, so there is an N with $p^N T^+ \subset A_j$. This proves the bounded-cokernel assertion with T^+ in place of S^+.

Let us identify the latter rings. Since $W \to V_j$ is surjective for large j, the valuation criterion for integral elements shows that $S_j^+/p^m \to T^+/p^m$ is injective. Given $f \in T^+$, first express $p^N f$ in A_j. Approximate its finitely many coefficients in R^+ by coefficients at a larger finite level, modulo p^{m+N}. We obtain $g \in S_{j'}^+$ whose image is congruent to $p^N f$. Surjectivity of $W \to V_{j'}$ implies $g \in p^N S_{j'}^+$, so division by p^N approximates f modulo p^m. Consequently

$$\varinjlim_j S_j^+/p^m = T^+/p^m.$$

Completion gives $S^+ = T^+$ and $S = T$, proving (i) and (ii).

For (iii), the integral approximation theorem for rational localizations of perfectoid algebras gives, for every $\epsilon > 0$, a topologically finitely generated R^+-subalgebra $T_\epsilon^+ \subset T^+$ with $p^\epsilon T^+ \subset T_\epsilon^+$. This is the rational-localization estimate of [**Sch12**, Lemma 6.4]; its generators are sufficiently deep p-power roots of the defining bounded fractions. In the finite étale case the same assertion follows from the uniformly almost finite generation in almost purity. Approximate the finitely many generators of T_ϵ^+ at a common finite level. The bounded error lies in $p^N T^+ \subset A_j$, and hence, after increasing j, all those generators lie in A_j. It follows that $p^\epsilon T^+ \subset A_j$. For a composite of finitely many maps, apply the estimates with positive exponents whose sum is ϵ. This is the approximation argument of [**Sch13b**, Lemma 4.5]. □

Lemma 5.18. *Let $U \in X_{proét}$ be perfectoid. For any $V \to U$ pro-étale, V is also perfectoid.*

Proof. The assertion is local on U and V, so assume that U is affinoid perfectoid and factor a presentation of $V \to U$ as an étale map followed by an inverse limit of finite étale surjections. Locally, the étale map is a composite of a finite étale map and a rational localization, so the preceding lemma handles it. Each finite étale stage of the remaining tower is then affinoid perfectoid by almost purity. The completion of their filtered direct limit is again perfectoid: the integral rings remain bounded, and surjectivity of Frobenius modulo a pseudo-uniformizer follows by representing an element at one finite stage and

taking its root there. The integral ring is the completion of the integral direct limit. This constructs affinoid perfectoid neighbourhoods covering V, as required. □

Corollary 5.19. *Let X be smooth over $\mathrm{Spa}(K, K^+)$. Then the set of $U \in X_{pro\acute{e}t}$ which are affinoid perfectoid forms a basis for the topology.*

Proof. If $X = \mathbf{T}^n$, then $\tilde{\mathbf{T}}^n \in X_{\mathrm{pro\acute{e}t}}$ is an explicit cover of X by affinoid perfectoids. Anything pro-étale over $\mathbf{T}^n$ again has a basis of affinoid perfectoid spaces by the last lemma, giving the claim: Indeed, any $U \in X_{\mathrm{pro\acute{e}t}}$ is covered by $U \times_{\mathbf{T}^n} \tilde{\mathbf{T}}^n$.

A smooth space X locally admits an étale map to $\mathbf{T}^n$. Pulling the preceding construction back along these charts gives the desired basis on X. □

We used smoothness here to get our hands on an explicit cover. In fact, a rather stronger result due to Colmez shows that the same assertion holds without smoothness.

Proposition 5.20. *Let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$. Then the set of $U \in X_{pro\acute{e}t}$ which are affinoid perfectoid forms a basis for the topology.*

Proof. It is enough to produce one affinoid perfectoid pro-étale cover of each connected affinoid open $\mathrm{Spa}(A, A^+)$. Once such a cover exists, its pullback to any other pro-étale object is perfectoid by the preceding lemma, and its affinoid opens give the required basis.

Here we use the following precise perfectoidization input of Colmez, in the form proved in [**Sch13b**, Proposition 4.8]: after extending the perfectoid ground field to its completed algebraic closure, iteratively adjoining p-th roots of every 1-unit, and taking the completed integral direct limit, produces a perfectoid affinoid algebra. The adjunctions are finite étale, since p and the units are invertible in the analytic algebra; the assertion that their completed limit is uniform and perfectoid is the substantive input. It does not follow merely from almost purity.

Order the root adjunctions in a transfinite tower, taking the appropriate preceding limit at limit stages. Adjoin the algebraic constants by the same convention. Each new finite collection of roots is a finite étale surjection, so the resulting map is an admissible cover. The perfectoidization input identifies its completed integral algebra as a perfectoid algebra. Finite étale covers preserve integral closures, so this completion is exactly the one in our definition of an affinoid perfectoid object. This supplies the desired cover. The use of the completed algebraic closure can be absorbed into the initial admissible tower of finite constant extensions. □

One interesting consequence of having these perfectoid objects around is that we can compute the étale cohomology of a connected noetherian affinoid with finite p-primary torsion coefficients using only its fundamental group. This is the sense in which the space is a $K(\pi, 1)$.

Theorem 5.21. *Let $X = \mathrm{Spa}(A, A^+)$ be any affinoid connected noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then for all finite p-primary torsion local systems $\mathbf{L}$ on $X_{\acute{e}t}$, the natural map*

$$H^i_{\mathrm{cont}}(\pi_1(X, \bar{x}), \mathbf{L}_{\bar{x}}) \to H^i(X_{\acute{e}t}, \mathbf{L})$$

is an isomorphism. Here $\bar{x}$ is a geometric point, and $\pi_1(X, \bar{x})$ is the profinite étale fundamental group.

PROOF. Let $f : X_{\text{ét}} \to X_{\text{fét}}$ be the natural morphism of sites. We have $f_*\mathbf{L} = \mathbf{L}$, and $R^i f_*\mathbf{L}$ is the sheafification of $U \mapsto H^i(U_{\text{ét}}, \mathbf{L})$ on the finite étale site. Thus it suffices to show that each positive-degree class is killed by a finite étale cover. After a finite étale cover we may trivialize $\mathbf{L}$; dévissage by powers of p then reduces to $\mathbf{L} = \mathbf{F}_p$.

Choose compatible geometric points on all connected finite étale covers of X, and let A_∞ be their filtered direct limit. Then A_∞ is connected and every faithfully flat finite étale A_∞-algebra has a section. Put A_∞^+ equal to the integral closure of A^+ and complete it p-adically. We use two precise limit results here. First, the Colmez perfectoidization input just used, applied inside this universal finite étale tower, shows that

$$(\widehat{A}_\infty, \widehat{A}_\infty^+)$$

is perfectoid: the tower contains all the successive root adjunctions and the algebraic constants, and its further finite étale stages remain perfectoid after completion by almost purity. Second, finite étale algebras descend through this completion and through the filtered tower, and torsion étale cohomology is continuous along the resulting perfectoid limit. These are [**Sch12**, Lemma 7.5 and Corollary 7.18]. In particular, for $X_\infty = \operatorname{Spa}(\widehat{A}_\infty, \widehat{A}_\infty^+)$,

$$H^j(X_{\infty,\text{ét}}, \mathbf{F}_p) = \varinjlim_i H^j(X_{i,\text{ét}}, \mathbf{F}_p),$$

and every finite étale cover of X_∞ has a section. These limit statements, as well as perfectoidization, are needed; almost purity alone is not a proof of them.

Tilt X_∞. Tilting identifies its étale site with that of the affinoid perfectoid $X_\infty^\flat$, and the latter has no nontrivial connected finite étale covers. On this characteristic-p site we have the exact Artin–Schreier sequence

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{O}_{X_\infty^\flat} \xrightarrow{\varphi - 1} \mathcal{O}_{X_\infty^\flat} \longrightarrow 0.$$

The structure sheaf of an affinoid perfectoid is acyclic on its étale site, by perfectoid Tate acyclicity and almost purity after inverting a pseudo-uniformizer. The sequence therefore gives vanishing in degrees at least two. In degree one, a section a defines the finite étale algebra $T^p - T - a$; a section of this cover supplies a root. Hence $\varphi - 1$ is surjective on global sections and degree-one cohomology also vanishes.

Continuity now says that every original cohomology class becomes zero at some finite stage. Thus $R^i f_*\mathbf{L} = 0$ for $i > 0$, and Leray identifies étale cohomology with finite étale cohomology. The latter is continuous cohomology of $\pi_1(X, \bar{x})$ with its action on $\mathbf{L}_{\bar{x}}$, proving the theorem. The same conclusion holds for discrete p-primary torsion local systems whenever they are expressed as filtered unions of finite local systems; cohomology on these qcqs sites commutes with that filtered union. □

LEMMA 5.22. *Let $U \in X_{\text{proét}}$ be affinoid perfectoid, with $\hat{U} = \operatorname{Spa}(R, R^+)$. We consider almost mathematics with respect to K^+ and the ideal of topologically nilpotent elements.*

(i) *For any nonzero $b \in K^+$, we have equalities $\mathcal{O}_X^+(U)/b = R^+/b =^a (\mathcal{O}_X^+/b)(U)$.*
(ii) *Let $b_1, b_2 \in K^+$ be nonzero elements with $|b_1| < |b_2|$. Then the image of $(\mathcal{O}_X^+/b_1)(U)$ in $(\mathcal{O}_X^+/b_2)(U)$ is R^+/b_2.*
(iii) *We have $\hat{\mathcal{O}}_X^+(U) = R^+$, and $\hat{\mathcal{O}}_X(U) = R$.*
(iv) *The ring $\hat{\mathcal{O}}_X^+(U)$ is the p-completion of $\mathcal{O}_X^+(U)$.*
(v) *The cohomology groups $H^i(U, \hat{\mathcal{O}}_X^+)$ are almost zero for $i > 0$.*

Thus, for all perfectoid $U \in X_{pro\acute{e}t}$, the locally v-ringed space $(|U|, \hat{\mathcal{O}}_X|_{|U|}, \{|\bullet(x)|\}_{x\in|U|})$ is a perfectoid space, given by $\hat{U}$.

Proof. For a nonzero $b \in K^+$, some power of p belongs to bK^+. Thus the construction of R^+ by p-adic completion gives $\mathcal{O}_X^+(U)/b = R^+/b$. We claim that

$$U \longmapsto (R^+/b)^a$$

is already an acyclic sheaf on the affinoid perfectoid basis. For a covering by affinoid perfectoids, quasicompactness reduces to finitely many members. The approximation lemma identifies its augmented Čech complex, in the almost category, with the filtered limit of the complexes for rational and finite étale coverings of perfectoid affinoids. The former are almost exact by perfectoid Tate acyclicity, and the latter by the almost faithfully flat descent supplied by almost purity. Filtered direct limits are exact. The same argument at the stages of an admissible tower, with compatible refinements at limit stages, gives the assertion for the corrected covering topology. Thus sheafification gives $(\mathcal{O}_X^+/b)^a$, and its higher cohomology on U vanishes. This proves (i) and the modulo-b form of (v).

For the stronger image assertion in (ii), set $c = b_1/b_2$, which is topologically nilpotent. Given $f \in (\mathcal{O}_X^+/b_1)(U)$, almost surjectivity in (i) gives $g \in R^+$ representing cf. Locally, g is divisible by c; the valuation criterion for R^+ then implies $g = ch$ with $h \in R^+$. Multiplication by c gives an injection

$$\mathcal{O}_X^+/b_2 \longrightarrow \mathcal{O}_X^+/b_1,$$

since these integral sheaves are scalar-torsion-free. It follows that f and h have the same image modulo b_2. Conversely, every element of R^+/b_2 lifts from R^+, proving (ii).

Sections commute with inverse limits. Part (ii), applied to consecutive powers of p, therefore gives

$$\hat{\mathcal{O}}_X^+(U) = \varprojlim_n (\mathcal{O}_X^+/p^n)(U) = \varprojlim_n R^+/p^n = R^+.$$

Inverting p proves (iii), and the definition of R^+ proves (iv). Finally, on the perfectoid basis the almost sections modulo p^n have surjective transition maps and the higher almost cohomology already vanishes. The almost version of Lemma 5.13 therefore proves (v). The equalities are compatible with rational restrictions, so they also identify the associated locally valued ringed space with $\hat{U}$. □

Lemma 5.23. *In the setting of the previous lemma, let $\mathbf{L}$ be an $\mathbf{F}_p$-local system on $X_{\acute{e}t}$. Then for all $i > 0$, the cohomology group*

$$H^i(U, \mathbf{L} \otimes \mathcal{O}_X^+/p)$$

is almost zero, and it is an almost finitely generated projective R^{+a}/p-module $M(U)$ for $i = 0$. Moreover, if $U' \in X_{pro\acute{e}t}$ is affinoid perfectoid corresponding to $\hat{U}' = \mathrm{Spa}(R', R'^+)$, and $U' \to U$ is some morphism in $X_{pro\acute{e}t}$, then $M(U') = M(U) \otimes_{R^{+a}/p} R'^{+a}/p$.

Proof. Choose a finite étale Galois cover $V \to U$ trivializing $\mathbf{L}$, with group G. Such a cover exists because the local system has finite rank and finite monodromy. By the preceding results V and all its iterated fibre products over U are affinoid perfectoid. Write $\hat{V} = \mathrm{Spa}(S, S^+)$. Almost purity says that S^{+a}/p is finite étale and faithfully flat over R^{+a}/p.

The module is free upstairs because the cover trivializes $\mathbf{L}$. The trivializations on the two pullbacks of V need not agree; their comparison is the descent datum, and on three pullbacks these comparisons satisfy the cocycle condition. Thus the module downstairs is obtained by imposing the same compatibility condition that appears when gluing sections of a sheaf. Faithfully flat descent identifies the downstairs module with the equalizer of the two pullback maps, and its augmented descent complex is exact; for ordinary modules, see [**The26**, Tags 023M and 023N]. Here we use the almost version, with almost faithful flatness supplied by almost purity.

The preceding lemma identifies the Čech complex of $\mathbf{L} \otimes \mathcal{O}_X^{+a}/p$ with this descent complex for $(S^{+a}/p)^r$. It is exact in positive degrees, and its degree-zero cohomology is an almost finite projective R^{+a}/p-module. The descent construction also commutes with base change: the base change of the descended module reconstructs the pulled-back descent datum, so uniqueness of descent identifies the two. Pulling the cover and its descent datum back to U' gives exactly

$$M(U') = M(U) \otimes_{R^{+a}/p} R'^{+a}/p,$$

which proves all the assertions. □

3. Finiteness of Cohomology

The calculations at the end of the last section were local: after tensoring a finite local system with the integral structure sheaf, we had almost no higher cohomology on our perfectoid objects. We would now like to see what this tells us about the cohomology of the whole space. In particular, we will prove finiteness and obtain our first comparison theorem, called the primitive comparison theorem.

THEOREM 5.24. *Let K be a complete algebraically closed extension of $\mathbf{Q}_p$ and $K^+ \subset K$ be an open and bounded valuation subring. Let X be a smooth proper adic space over $\mathrm{Spa}(K, K^+)$, and $\mathbf{L}$ an $\mathbf{F}_p$-local system on $X_{\text{ét}}$. Then*

(i) *$H^i(X_{\text{ét}}, \mathbf{L})$ is a finite-dimensional $\mathbf{F}_p$-vector space, which vanishes for $i > 2 \dim X$.*
(ii) *There is an isomorphism of almost K^+-modules*

$$H^i(X_{\text{ét}}, \mathbf{L}) \otimes K^{+a}/p \cong H^i(X_{\text{ét}}, \mathbf{L} \otimes \mathcal{O}_X^{+a}/p)$$

for all $i \geqslant 0$.

There is a convenient way to get at the local system in characteristic p: we can recover it from a sheaf with Frobenius by using the Artin–Schreier sequence. Since the objects we have been working with are perfectoid, we can use tilting to put ourselves in this situation. We first define the corresponding sheaf.

DEFINITION 5.25. Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. The tilted integral structure sheaf $\widehat{\mathcal{O}}^+_{X^\flat}$ is given by the inverse limit $\varprojlim_\Phi \mathcal{O}_X^+/p$, where the transition maps are the Frobenius. If X is defined over $\mathrm{Spa}(K, K^+)$, where K is a perfectoid field with open and bounded valuation ring K^+, we define $\widehat{\mathcal{O}}_{X^\flat} := \widehat{\mathcal{O}}^+_{X^\flat} \otimes_{K^{\flat +}} K^\flat$.

LEMMA 5.26. *Let K be a perfectoid field of characteristic zero with open and bounded valuation subring K^+. Let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$, and $U \in X_{\text{proét}}$ be affinoid perfectoid, with $\hat{U} = \mathrm{Spa}(R, R^+)$ where (R, R^+) is a perfectoid affinoid (K, K^+)-algebra. Let $(R^\flat, R^{\flat +})$ be its tilt. Then:*

(i) *We have* $\hat{\mathcal{O}}^+_{X^\flat}(U) = R^{\flat+}$ *and* $\hat{\mathcal{O}}_{X^\flat}(U) = R^\flat$.

(ii) *The cohomology groups* $H^i(U, \hat{\mathcal{O}}^+_{X^\flat})$ *are almost zero for* $i > 0$ *with respect to almost mathematics defined by* $K^{\flat+}$ *and its ideal of topologically nilpotent elements.*

Proof. On an affinoid perfectoid, Frobenius is surjective on R^+/p. The preceding structure-sheaf lemma identifies the inverse limit of the section groups along Frobenius with $\varprojlim_\varphi R^+/p = R^{\flat+}$; its sharper image assertion removes the almost error in the section groups. Inverting a pseudo-uniformizer gives the second equality in (i).

For (ii), one can repeat the almost Čech argument on the perfectoid basis after tilting. Rational localizations and finite étale covers tilt, their integral descent complexes are almost exact, and their completed inverse limits have the same property. Thus $R^{\flat+}/\varpi^m$ has no higher almost cohomology on U for a pseudo-uniformizer $\varpi \in K^{\flat+}$. The transition maps on these section groups are surjective. Applying the almost version of Lemma 5.13 to this tower and using $R^{\flat+} = \varprojlim_m R^{\flat+}/\varpi^m$ gives the desired almost vanishing. Notice that this invokes a verified acyclicity and Mittag–Leffler condition, not an unrestricted assertion that inverse limits of sheaves are exact. □

Before giving the proof, let us isolate the two substantial inputs. They are weaker than the finiteness and comparison assertions we want to establish, and separating them makes the argument clearer.

(i) *Almost finiteness.* If Y is proper smooth over an algebraically closed perfectoid field C of characteristic zero and $\mathbf{L}$ is a finite-rank $\mathbf{F}_p$-local system, then

$$H^j(Y_{\text{ét}}, \mathbf{L} \otimes \mathcal{O}^+_Y/p)$$

is almost finitely generated over $\mathcal{O}_C$, and is almost zero for $j > 2 \dim Y$. This is [**Sch13b**, Lemma 5.8]. The proof uses the integral toric-cover estimates, not finiteness of $H^j(Y, \mathbf{L})$: nested finite rational covers give maps of Čech spectral sequences whose images are almost finitely generated. Taking $j + 2$ such covers controls every term contributing to total degree j. The vanishing bound combines the local bound $\dim Y$ with the cohomological dimension $\dim Y$ of the underlying analytic topology. These integral estimates are the geometric input we import here.

(ii) *Frobenius and almost freeness.* Let C be algebraically closed of characteristic p, let $\pi \in \mathcal{O}_C$ be a pseudo-uniformizer, and let M_k be $\mathcal{O}_C/\pi^k$-modules. Suppose M_1 is almost finitely generated, there are maps

$$p_k : M_{k+1} \to M_k, \qquad q_k : M_k \to M_{k+1}, \qquad p_k q_k = \pi,$$

the sequence $M_1 \xrightarrow{q_k \cdots q_1} M_{k+1} \xrightarrow{p_k} M_k$ is exact in the middle, and there are compatible Frobenius-semilinear isomorphisms

$$M_k \otimes_{\mathcal{O}_C/\pi^k, \varphi} \mathcal{O}_C/\pi^{pk} \xrightarrow{\sim} M_{pk}.$$

Then, for one finite integer r, there are compatible isomorphisms

$$M_k^a \cong (\mathcal{O}_C^a/\pi^k)^r$$

carrying p_k to reduction, q_k to multiplication by π, and Frobenius to coordinate-wise Frobenius. This is [**Sch13b**, Lemma 2.12].

This last conclusion is quite a bit stronger than just knowing that the modules are almost finitely generated. Let us see where the extra information comes from. The classification of almost finitely generated torsion modules over a valuation ring assigns their lengths a decreasing sequence of real invariants. Exactness gives the inequality that the invariant of M_k is at most k times that of M_1, whereas Frobenius multiplies it by p. Comparing these inequalities for arbitrarily large p-powers forces equality. Consequently the displayed middle-exact sequences are almost short exact, and their inverse limit M^a is flat, complete, and almost finitely generated, with $M^a/\pi^k = M_k^a$. This valuation-module classification is part of the algebraic input in Lemma 2.12.

We can now invert π and study the Frobenius action on a finite-dimensional vector space. The useful fact here is that we can choose a basis fixed by Frobenius. To see this last step directly, write $\Phi(v) = Av^{(p)}$ in some basis. The map $v \mapsto Av^{(p)} - v$ is finite étale of degree p^r: its derivative is -1, and its defining equations express each p-th power as a linear combination of the coordinates. Over the algebraically closed field its kernel has p^r elements, hence dimension r over $\mathbf{F}_p$. An $\mathbf{F}_p$-independent list of fixed vectors is also linearly independent over C. Indeed, applying Frobenius to a shortest nontrivial linear relation, normalized to have one coefficient equal to 1, shows that every coefficient is fixed by Frobenius, a contradiction. The fixed vectors therefore give the desired basis.

The Frobenius-fixed basis also gives us information about the integral lattice. In such a basis the image M' of M is bounded between $\pi^N \mathcal{O}_C^r$ and $\pi^{-N}\mathcal{O}_C^r$. Frobenius invariance improves these bounds to $\pi^{N/p^m}\mathcal{O}_C^r$ and $\pi^{-N/p^m}\mathcal{O}_C^r$ for every m. Their difference disappears in the almost category. Thus $M'^a = (\mathcal{O}_C^a)^r$, compatibly with Frobenius, which explains both the common rank and the normalized basis.

Proof of Theorem 5.24. First consider $K^+ = \mathcal{O}_K$, and write $d = \dim X$. Choose $\pi \in \mathcal{O}_{K^\flat}$ with $\pi^\sharp = p$, and put

$$\mathcal{F}_k = \mathbf{L} \otimes \widehat{\mathcal{O}}^+_{X^\flat}/\pi^k, \qquad M_k^i = H^i(X_{\text{pro\'et}}, \mathcal{F}_k).$$

The tilted integral sheaf is a sheaf of perfect, scalar-torsion-free $\mathcal{O}_{K^\flat}$-algebras. Since a torsion-free module over a valuation ring is flat, its coefficient sequences give maps p_k and q_k on cohomology with precisely the middle-exactness required in the second input. Frobenius and its flat scalar extension give the indicated semilinear isomorphisms $M_k^i \otimes_\varphi \mathcal{O}_{K^\flat}/\pi^{pk} \cong M_{pk}^i$. Finally,

$$\widehat{\mathcal{O}}^+_{X^\flat}/\pi \cong \mathcal{O}_X^+/p$$

on the perfectoid basis, and hence as sheaves. The first input, together with the comparison between the étale and pro-étale sites, proves the required almost finite generation of M_1^i. We may therefore apply the second input to obtain

$$(M_k^i)^a \cong (\mathcal{O}^a_{K^\flat}/\pi^k)^{r_i},$$

compatibly with all reduction maps and Frobenius. The integer r_i is finite, and is zero for $i > 2d$, since then $(M_1^i)^a = 0$.

We next justify passage to the inverse limit. On the affinoid perfectoid basis trivializing $\mathbf{L}$, the sheaves $\mathcal{F}_k^a$ are acyclic and their section maps are surjective. Lemma 5.13 therefore gives

$$R\varprojlim_k \mathcal{F}_k^a = (\mathbf{L} \otimes \widehat{\mathcal{O}}^+_{X^\flat})^a.$$

On global cohomology the preceding compatible free descriptions also have surjective transition maps. Thus the $R^1\varprojlim$ terms in the cohomology-limit sequence vanish, and

$$H^i(X_{\text{proét}}, \mathbf{L} \otimes \widehat{\mathcal{O}}^+_{X^\flat})^a \cong (\mathcal{O}^a_{K^\flat})^{r_i}.$$

Inverting π removes the almost qualifier and gives a Frobenius-compatible isomorphism

$$H^i(X_{\text{proét}}, \mathbf{L} \otimes \widehat{\mathcal{O}}_{X^\flat}) \cong (K^\flat)^{r_i}.$$

Here cohomology commutes with this filtered localization on the qcqs site.

The Artin–Schreier sequence

$$0 \longrightarrow \mathbf{L} \longrightarrow \mathbf{L} \otimes \widehat{\mathcal{O}}_{X^\flat} \xrightarrow{\varphi-1} \mathbf{L} \otimes \widehat{\mathcal{O}}_{X^\flat} \longrightarrow 0$$

is exact on $X_{\text{proét}}$. To check surjectivity, trivialize $\mathbf{L}$ on an affinoid perfectoid and adjoin a root of $T^p - T - a$ on its tilt. This is finite étale, untilts to a finite étale cover, and descends to a finite stage of the pro-étale presentation. Its kernel is the locally constant sheaf $\mathbf{F}_p$, coordinate by coordinate. Since $K^\flat$ is algebraically closed, coordinatewise $\varphi - 1$ is surjective on $(K^\flat)^{r_j}$ for every j. The cohomology sequence, in degrees $i-1$ and i, consequently identifies

$$H^i(X_{\text{ét}}, \mathbf{L}) = H^i(X_{\text{proét}}, \mathbf{L}) \cong \left((K^\flat)^{r_i}\right)^{\varphi=1} = \mathbf{F}_p^{r_i}.$$

This proves finiteness and the stated vanishing.

It also proves the comparison map, not merely a dimension equality. In the Frobenius-fixed basis, the natural integral map

$$H^i(X_{\text{ét}}, \mathbf{L}) \otimes \mathcal{O}^a_{K^\flat} \longrightarrow H^i(X_{\text{proét}}, \mathbf{L} \otimes \widehat{\mathcal{O}}^+_{X^\flat})^a$$

identifies the fixed basis vectors on both sides, so is an isomorphism. The compatible descriptions of M_k^i show that reduction modulo π gives the corresponding cohomology modulo π. Identifying this quotient with $\mathcal{O}^+_X/p$ yields exactly the natural map in (ii).

For a general open bounded valuation subring K^+, put $X' = X \times_{\mathrm{Spa}(K,K^+)} \mathrm{Spa}(K, \mathcal{O}_K)$. On an affinoid perfectoid cover, replacing the chosen integral subring by the ring of power-bounded elements changes it by an almost zero quotient: multiplication by any topologically nilpotent scalar carries the latter ring into the former. Lemma 5.23, and its tilted version, therefore identify the coefficient cohomology of X and X' almost isomorphically. A perfectoid hypercover and its cohomology spectral sequence give the same identification globally. Apply the preceding argument to these coefficient groups; the Artin–Schreier sequence is still the sequence on X itself. It recovers $H^i(X_{\text{ét}}, \mathbf{L})$ and the natural integral comparison exactly as before, now in the almost category defined by K^+. □

4. Period Sheaves

The primitive comparison already relates étale cohomology to the integral structure sheaf. Our goal, however, is to compare with de Rham cohomology, so we will need sheaves on the pro-étale site that play the role of Fontaine's period rings. In addition to the analogue of B_{dR} itself, we will need a sheaf on which we can take differentials. It is this additional sheaf that will let us use the de Rham complex.

Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Recall that we have defined the structure sheaf $\widehat{\mathcal{O}}_X^+$ on $X_{\mathrm{pro\acute{e}t}}$.

Definition 5.27. We define the following sheaves on $X_{\mathrm{pro\acute{e}t}}$.

- We define $\mathbf{A}_{\mathrm{inf}} = W(\widehat{\mathcal{O}}^+_{X^\flat})$, and its rational version $\mathbf{B}_{\mathrm{inf}} = \mathbf{A}_{\mathrm{inf}}[p^{-1}]$. There is a map $\theta : \mathbf{A}_{\mathrm{inf}} \to \widehat{\mathcal{O}}_X^+$, which extends to a map $\mathbf{B}_{\mathrm{inf}} \to \widehat{\mathcal{O}}_X$, which by abuse of notation we also call θ.
- The *positive de Rham sheaf* is $\mathbf{B}^+_{\mathrm{dR}} := \varprojlim \mathbf{B}_{\mathrm{inf}}/(\ker\theta)^n$. Moreover, there is a filtration, given by $\mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} = (\ker\theta)^i \mathbf{B}^+_{\mathrm{dR}}$.
- The *de Rham sheaf* is $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[t^{-1}]$ where t is any local generator of $\mathrm{Fil}^1 \mathbf{B}^+_{\mathrm{dR}}$. It also has a filtration, given by

$$\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} = \sum_{j\in\mathbf{Z}} t^{-j} \mathrm{Fil}^{i+j} \mathbf{B}^+_{\mathrm{dR}}$$

We will informally refer to these as *period sheaves*.

If K is a perfectoid field of characteristic zero, with valuation subring $K^+ \subset K$, and (R, R^+) is a perfectoid affinoid (K, K^+)-algebra, we define:

- $\mathbf{A}_{\mathrm{inf}}(R, R^+) = W(R^{\flat+})$
- $\mathbf{B}_{\mathrm{inf}}(R, R^+) = \mathbf{A}_{\mathrm{inf}}(R, R^+)[p^{-1}]$
- $\mathbf{B}^+_{\mathrm{dR}}(R, R^+) = \varprojlim \mathbf{B}_{\mathrm{inf}}(R, R^+)/(\ker\theta)^i$

Lemma 5.28. *There is an element $\xi \in \mathbf{A}_{\mathrm{inf}}(K, K^+)$ that generates $\ker\theta$, where θ is the map $\mathbf{A}_{\mathrm{inf}}(K, K^+) \to K^+$. The element ξ is a nonzerodivisor, and is hence unique up to multiplication by a unit. Moreover, this remains true when (K, K^+) is replaced by any affinoid perfectoid (K, K^+)-algebra (R, R^+).*

Proof. Choose $\pi \in K^{\flat+}$ such that $\pi^\sharp = pv$ with $v \in (K^+)^\times$. Perfectoidness gives

$$R^{\flat+}/\pi \xrightarrow{\sim} R^+/p.$$

This first proves surjectivity of θ: lift a target element modulo p by a Teichmüller element, subtract it, divide by p, and repeat. The resulting Witt series converges. Choose $a \in W(K^{\flat+})$ with $\theta(a) = v$, and set $\xi = [\pi] - pa$. Then $\theta(\xi) = 0$ and $\xi \bmod p = \pi$.

For $x \in \ker\theta \subset W(R^{\flat+})$, the displayed isomorphism gives $x \bmod p \in \pi R^{\flat+}$. We can therefore write $x = \xi y_0 + p x_1$. Since R^+ has no p-torsion, $\theta(x_1) = 0$. Repeating gives

$$x = \xi(y_0 + p y_1 + \cdots + p^{m-1} y_{m-1}) + p^m x_m.$$

The Witt ring is p-adically complete, so x is divisible by ξ. If $\xi y = 0$, reduction modulo p gives $\pi\bar{y} = 0$. Multiplication by the nonzero scalar π is injective on $R^{\flat+}$, hence $y \in pW(R^{\flat+})$. Repeating and using p-adic separatedness gives $y = 0$. Thus the same scalar ξ generates the kernel for every (R, R^+) and is a nonzerodivisor. Two nonzerodivisor generators of a principal ideal differ by a unit, proving the final assertion. □

Thus, we may also define $\mathbf{B}_{\mathrm{dR}}(R, R^+) = \mathbf{B}^+_{\mathrm{dR}}(R, R^+)[\xi^{-1}]$, with filtration given by $\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}}(R, R^+) = \xi^i \mathbf{B}^+_{\mathrm{dR}}(R, R^+)$.

Corollary 5.29. *For all integers $i \in \mathbf{Z}$, we have $\mathrm{gr}^i \mathbf{B}_{\mathrm{dR}}(R, R^+) \cong \xi^i R$.*

Proof. The lemma gives $\mathbf{B}_{\mathrm{inf}}(R, R^+)/\xi = R$. Since ξ is a nonzerodivisor, multiplication by ξ^i identifies each successive quotient ξ^i/ξ^{i+1} with the degree-zero quotient. Completion preserves these quotients, so $\mathbf{B}^+_{\mathrm{dR}}(R, R^+)/\xi = R$. After inverting ξ, the same calculation applies for negative i. Thus

$$\mathrm{Fil}^i / \mathrm{Fil}^{i+1} = \xi^i \mathbf{B}^+_{\mathrm{dR}} / \xi^{i+1} \mathbf{B}^+_{\mathrm{dR}} \cong \xi^i R.$$

The notation records a choice of generator; intrinsically the later Tate-twist description is independent of that choice. □

Fix some $\pi \in K^\flat$ such that $\pi^\sharp/p \in (K^+)^\times$. All of the rings thus far are $\mathbf{A}_{\mathrm{inf}}(K, K^+)$-algebras via the structural map from the ground field; on the structure sheaves this action factors through θ. We now consider almost mathematics with respect to this ring, and the ideal generated by $[\pi^{1/p^N}]$.

Theorem 5.30. *Let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$. Let $U \in X_{pro\acute{e}t}$ be affinoid perfectoid, with $\widehat{U} = \mathrm{Spa}(R, R^+)$.*

(i) *There is a canonical isomorphism*

$$\mathbf{A}_{\mathrm{inf}}(U) \cong \mathbf{A}_{\mathrm{inf}}(R, R^+).$$

The analogous assertion holds for $\mathbf{B}_{\mathrm{inf}}$, $\mathbf{B}^+_{\mathrm{dR}}$, and $\mathbf{B}_{\mathrm{dR}}$.

(ii) *The cohomology groups $H^i(U, \mathcal{F})$ are almost zero for all $i > 0$, where $\mathcal{F}$ is any of the sheaves above.*

(iii) *The element $[\pi]$ becomes invertible in $\mathbf{B}^+_{\mathrm{dR}}(U)$ and $\mathbf{B}_{\mathrm{dR}}(U)$. Thus the cohomology groups $H^i(U, \mathcal{F})$ vanish for these sheaves when $i > 0$.*

Proof. The tilted structure-sheaf lemma identifies the sections of $\widehat{\mathcal{O}}^+_{X^\flat}$ with $R^{\flat +}$ and makes their higher cohomology almost zero. The successive kernels of $W_m \to W_{m-1}$ are Frobenius twists of this additive sheaf. Induction on m therefore gives the section formula and almost acyclicity for W_m. Their transition maps on sections are surjective; Lemma 5.13, in the almost category, gives the same conclusions for $\mathbf{A}_{\mathrm{inf}} = \varprojlim_m W_m$. On this qcqs object cohomology commutes with filtered localization, so inverting p gives the result for $\mathbf{B}_{\mathrm{inf}}$.

By Lemma 5.28, the quotients by powers of $\ker\theta$ are the quotients by ξ^m. Their successive quotients are $\widehat{\mathcal{O}}_X$, whose positive-degree cohomology vanishes after inverting p. The exact sequences for these quotients give their section formula and acyclicity. One can equally start with the almost assertions for $\mathbf{B}_{\mathrm{inf}}$: the image of $[\pi]$ modulo ξ is the unit $\pi^\sharp$ in R, so $[\pi]$ is a unit modulo every ξ^m. Thus almost isomorphisms over these quotients are genuine isomorphisms.

The maps between the section rings modulo ξ^m are surjective. Applying Lemma 5.13 once more proves the claims for $\mathbf{B}^+_{\mathrm{dR}}$. In this complete ring, $[\pi]$ is invertible because its reduction modulo ξ is a unit; lifting the inverse is the convergent geometric-series argument. Finally, localizing at ξ gives the assertions for $\mathbf{B}_{\mathrm{dR}}$. This proves (i)–(iii), and in particular explains why the last two sheaves are actually acyclic rather than only almost acyclic. □

We will soon be computing cohomology using towers of finite étale covers. On the repeated fiber products of such a cover, we will need to describe sections of the period sheaves. Here is a useful calculation for the case where we have copies of a perfectoid object parametrized by a profinite set.

COROLLARY 5.31. *Let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$, and let $U \in X_{pro\acute{e}t}$ be affinoid perfectoid. Let S be a profinite set, and set $V := S \times U$, which is again affinoid perfectoid. Then for all period sheaves $\mathcal{F}$, we have the equality*

$$\mathcal{F}(V) = \mathrm{Hom}_{\mathrm{cont}}(S, \mathcal{F}(U))$$

Here, $\mathcal{F}$ is allowed to be any of:

$$\widehat{\mathcal{O}}_X, \widehat{\mathcal{O}}_X^+, \widehat{\mathcal{O}}_{X^\flat}, \widehat{\mathcal{O}}_{X^\flat}^+, \mathbf{A}_{\mathrm{inf}}, \mathbf{B}_{\mathrm{inf}}, \mathbf{B}_{\mathrm{dR}}^+, \mathbf{B}_{\mathrm{dR}}, \mathrm{gr}^i \mathbf{B}_{\mathrm{dR}}$$

PROOF. Write $S = \varprojlim S_\alpha$ with S_α finite. The integral algebra of $S_\alpha \times U$ is the finite product $(R^+)^{S_\alpha}$. The completed direct limit of these products is the ring of continuous maps $S \to R^+$: modulo any power of a pseudo-uniformizer, a continuous map factors through a finite quotient of the compact space S. Thus V is affinoid perfectoid and the assertion holds for $\widehat{\mathcal{O}}_X^+$, and likewise for the tilted integral sheaf.

The remaining identities follow by the operations used to construct the period sheaves. Finite Witt vectors are given by finitely many coordinates, and infinite Witt vectors by their inverse limit. Inverting p allows a common denominator for a continuous map on a compact source, and completion along ξ is checked quotient by quotient. The same argument applies after inverting ξ and to the graded quotients. Here we give the integral rings their adic topology, finite Witt vectors the topology of their coordinates, and the de Rham rings their quotient and inverse-limit topologies; localizations carry the corresponding inductive-limit topology. These are the topologies meant by $\mathrm{Hom}_{\mathrm{cont}}$. All identifications commute with restrictions, so they give the stated equality of sections for every listed sheaf. □

PROPOSITION 5.32. *Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. For all $i \in \mathbf{Z}$, we have $\mathrm{gr}^i \mathbf{B}_{\mathrm{dR}} \cong \widehat{\mathcal{O}}_X(i)$, where (i) denotes the Tate twist.*

PROOF. Let K be the completion of $\mathbf{Q}_p(\mu_{p^\infty})$. A compatible system of primitive p-power roots of unity gives an element $\epsilon \in \mathcal{O}_{K^\flat}$. We have an element $t = \log([\epsilon]) \in \mathrm{Fil}^1 \mathbf{B}_{\mathrm{dR}}(K, K^+)$ which generates Fil^1. This gives an isomorphism $\mathrm{gr}^i \mathbf{B}_{\mathrm{dR}} \cong t^i \widehat{\mathcal{O}}_X$ on $X_{\mathrm{pro\acute{e}t}}/X_K$. But the action of the Galois group $\mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ on t is via the cyclotomic character, so the isomorphism descends to an isomorphism $\mathrm{gr}^i \mathbf{B}_{\mathrm{dR}} \cong \widehat{\mathcal{O}}_X(i)$ on $X_{\mathrm{pro\acute{e}t}}$. □

So far we have constructed a sheaf $\mathbf{B}_{\mathrm{dR}}$ much as we constructed the period ring in Chapter 2. There is something missing, though. We would like to be able to work with the differential forms on X, which means we need a period sheaf on which the functions on X act and on which we can differentiate them. We therefore construct a second kind of period sheaf, an $\mathcal{O}_X$-algebra with a connection.

DEFINITION 5.33. Let k be a discretely valued complete nonarchimedean extension of $\mathbf{Q}_p$, with perfect residue field κ, and X a smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$.

- The *sheaf of differentials* is $\Omega_X^1 = \nu^* \Omega_{X_{\acute{e}t}}^1$, with higher exterior powers Ω_X^i.

- The *structural infinitesimal sheaf* is the tensor product $\mathcal{O}\mathbf{B}_{\mathrm{inf}} := \mathcal{O}_X \otimes_{W(\kappa)} \mathbf{B}_{\mathrm{inf}}$, where $W(\kappa) = \nu^* W(\kappa)$ is the constant sheaf associated to $W(\kappa)$. We again have a map $\theta\colon \mathcal{O}\mathbf{B}_{\mathrm{inf}} \to \widehat{\mathcal{O}}_X$.
- To define the *positive structural de Rham sheaf*, an additional completion is essential. On an affinoid perfectoid $U = \varprojlim U_i$, with $U_i = \mathrm{Spa}(R_i, R_i^+)$ and $\widehat{U} = \mathrm{Spa}(R, R^+)$, put

$$C_i(U) = \left(R_i^+ \widehat{\otimes}_{W(\kappa)} \mathbf{A}_{\mathrm{inf}}(R, R^+)\right)[1/p].$$

  Here the completed tensor product is the p-adic completion of the integral tensor product, before inverting p. The map $\theta : C_i(U) \to R$ multiplies the structural map $R_i^+ \to R^+$ with the usual map on $\mathbf{A}_{\mathrm{inf}}$. Define $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$ to be the sheafification of

$$U \longmapsto \varinjlim_i \varprojlim_m C_i(U)/(\ker\theta)^m.$$

  The raw tensor product $\mathcal{O}\mathbf{B}_{\mathrm{inf}}$ maps naturally to this sheaf. Its filtration is the completed $\ker\theta$-adic filtration. This is the corrected definition of [Sch16]; completing only the raw rational tensor product along $\ker\theta$ does not give the required sheaf.
- The *structural de Rham sheaf* is $\mathcal{O}\mathbf{B}_{\mathrm{dR}} := \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+[t^{-1}]$, where t is a generator of $\mathrm{Fil}^1 \mathbf{B}_{\mathrm{dR}}^+$. It also has a filtration, given by

$$\mathrm{Fil}^i \mathcal{O}\mathbf{B}_{\mathrm{dR}} = \sum_{j \in \mathbf{Z}} t^{-j} \mathrm{Fil}^{i+j} \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$$

We can now extend the usual differential on the functions on X to our structural period sheaf. There is a canonical map $d\colon \mathcal{O}_X \to \Omega^1_X$, given by sending $f \mapsto df$. The sheaf $\mathcal{O}\mathbf{B}_{\mathrm{inf}}$ admits a unique $\mathbf{B}_{\mathrm{inf}}$-linear connection

$$\nabla\colon \mathcal{O}\mathbf{B}_{\mathrm{inf}} \to \mathcal{O}\mathbf{B}_{\mathrm{inf}} \otimes_{\mathcal{O}_X} \Omega^1_X$$

, extending the map d. The derivation is continuous on the integral tensors just used, and hence extends through their p-adic completions and their $\ker\theta$-adic completions to a connection

$$\nabla\colon \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega^1_X$$

, which is $\mathbf{B}_{\mathrm{dR}}^+$-linear. Finally, since $t \in \mathbf{B}_{\mathrm{dR}}$, we may further extend this to a connection

$$\nabla\colon \mathcal{O}\mathbf{B}_{\mathrm{dR}} \to \mathcal{O}\mathbf{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega^1_X$$

It is not yet very clear from this construction what sections of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$ actually look like, or how the connection acts on them. Let us work this out locally. Fix an algebraic extension of k whose completion K is perfectoid, and write X_K for the base change of X to $\mathrm{Spa}(K, \mathcal{O}_K)$, also considered as an object of $X_{\mathrm{pro\acute{e}t}}$. Choose an étale chart $X \to \mathbf{T}^n$ and let $\tilde{X} = X \times_{\mathbf{T}^n} \tilde{\mathbf{T}}^n$. After extending scalars to K, we get a perfectoid object $\tilde{X}_K \in X_{\mathrm{pro\acute{e}t}}/X_K$.

Notice that in the structural period ring we have two versions of each of our toric coordinates. Taking their differences gives us a particularly nice set of local coordinates in which to study the connection. On the localized site $X_{\mathrm{pro\acute{e}t}}/\tilde{X}$, we have elements $u_i = T_i \otimes 1 - 1 \otimes [T_i^\flat] \in \mathcal{O}\mathbf{B}_{\mathrm{inf}}|_{\tilde{X}} := (\mathcal{O}_X \otimes_{W(\kappa)} W(\widehat{\mathcal{O}}^+_{X^\flat}))|_{\tilde{X}}$. The u_i are all killed by θ, where $T_i^\flat \in \widehat{\mathcal{O}}^+_{X^\flat} = \varprojlim \mathcal{O}_X^+/p$ is given by the sequence $(T_i, T_i^{1/p}, \ldots)$.

PROPOSITION 5.34. *The map*

$$\mathbf{B}^+_{\mathrm{dR}}|_{\tilde{X}}[[X_1,\dots,X_n]] \to \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}|_{\tilde{X}}$$

given by sending $X_i \mapsto u_i$ is an isomorphism of sheaves over $X_{pro\acute{e}t}/\tilde{X}$.

PROOF. We work on an affinoid perfectoid $U \to \widetilde{X}_K$ and use the notation $C_i(U)$ from the corrected definition. Write $B^+ = \mathbf{B}^+_{\mathrm{dR}}(R, R^+)$ and $S_i = \varprojlim_m C_i(U)/(\ker\theta)^m$. The proposed map $B^+[[X_1,\dots,X_n]] \to S_i$ sends X_j to u_j and is continuous for the ideals $(\xi, X_1,\dots,X_n)$ and $\ker\theta$.

For sufficiently large i, the étale toric chart supplies a map in the other direction. Send T_j to $[T_j^{\flat}] + X_j$. These choices describe the map on the base of the chart. To extend it to the chart itself, we use formal étaleness: once the base map is fixed, a map into a quotient by a nilpotent ideal has a unique lift. Locally this is the Newton correction for a polynomial with invertible derivative; standard étale presentations give the general case [**The26**, Tags 00UB and 00UE]. In the present chart, this lifts the map on R_i uniquely across the successive nilpotent thickenings

$$B^+[[X_1,\dots,X_n]]/(\xi, X_1,\dots,X_n)^m,$$

compatibly as m varies. On the integral rings the maps are bounded; therefore they extend to the p-adically completed tensors defining $C_i(U)$. Passing to the limit gives $S_i \to B^+[[X_1,\dots,X_n]]$. Its composite with the proposed map is the identity on coefficients and on all X_j.

For completeness, the other composite requires the integral completion in our definition. The completed infinitesimal diagonal of the smooth toric chart is generated, modulo its m-th power, by the differences $T_j \otimes 1 - 1 \otimes T_j$, up to bounded p-power torsion on an integral model. This follows from the truncated polynomial description of the diagonal of the torus and formal étale base change; clearing denominators on the finite presentation gives a single bound for each fixed m. Tensoring this integral description with $\mathbf{A}_{\mathrm{inf}}(R, R^+)$, completing p-adically, and then inverting p removes that bounded torsion. It follows that the images of B^+ and the u_j generate $C_i(U)/(\ker\theta)^m$. Both composites fix these generators, and hence are the identity modulo every m. Separated completeness proves they are inverse. This is precisely the corrected diagonal argument in [**Sch16**], strengthening [**Sch13b**, Proposition 6.10].

The result is independent of all sufficiently large finite levels, is compatible with restrictions in U, and already defines a sheaf. Sheafification therefore gives the asserted isomorphism. □

With this description, the connection is just differentiation in the formal variables. We are now in a situation where we can integrate term by term, much as in the usual proof of the Poincaré lemma. In this way we can relate the absolute period sheaf to a complex involving differential forms on X.

COROLLARY 5.35 (Poincaré Lemma). *Let X be a smooth n-dimensional adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Then the following sequence of sheaves on $X_{pro\acute{e}t}$ is exact.*

$$0 \to \mathbf{B}_{\mathrm{dR}} \to \mathcal{O}\mathbf{B}_{\mathrm{dR}} \to \mathcal{O}\mathbf{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega^1_X \to \cdots \to \mathcal{O}\mathbf{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega^{\dim X}_X \to 0$$

Moreover, ∇ satisfies Griffiths transversality with respect to the filtration on $\mathcal{O}\mathbf{B}^+_{\mathrm{dR}}$.

PROOF. The assertion is local on the pro-étale site. On a toric perfectoid cover, Proposition 5.34 identifies the connection with formal differentiation in the variables $u_1, \dots, u_n$, with coefficients in $\mathbf{B}_{\mathrm{dR}}^+$. The formal de Rham complex over a $\mathbf{Q}$-algebra is exact in positive degrees: on a homogeneous polynomial differential form of polynomial degree a and form degree q with $a + q > 0$, contraction with the Euler vector field, divided by $a + q$, is a contracting homotopy. Apply this term by term to a formal power series. Its degree-zero kernel consists precisely of the constant coefficients $\mathbf{B}_{\mathrm{dR}}^+$.

This homotopy is compatible with the completed filtration: multiplication by a variable raises its order by one, exactly compensating for the change in form degree. Thus the corresponding filtered complexes and their graded complexes are exact as well. Inverting t gives the displayed sequence. Since differentiation takes the ideal $(\xi, u_1, \dots, u_n)^m$ into its $(m-1)$-st power times Ω_X^1, it also proves Griffiths transversality. □

COROLLARY 5.36. *Let X be a smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Then we have a short exact sequence*

$$0 \to \widehat{\mathcal{O}}_X(1) \to \mathrm{gr}^1 \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X \otimes_{\mathcal{O}_X} \Omega_X^1 \to 0$$

of sheaves on $X_{pro\acute{e}t}$.

PROOF. On a toric perfectoid chart, $\ker \theta$ is generated by the regular parameters $\xi, u_1, \dots, u_n$. Modulo its square, the classes of the u_j map under the connection to the basis dT_j of Ω_X^1, while ξ maps to zero. Its kernel is therefore $\xi\widehat{\mathcal{O}}_X = \widehat{\mathcal{O}}_X(1)$. This proves exactness locally. The maps are the structural inclusion and the connection, so the local sequences glue to the displayed canonical sequence, usually called Faltings's extension. □

COROLLARY 5.37. *On a toric chart as in Proposition 5.34, and after choosing ξ, for any $i \in \mathbf{Z}$ we have an isomorphism of sheaves*

$$\mathrm{gr}^i \mathcal{O}\mathbf{B}_{\mathrm{dR}} \cong \xi^i \widehat{\mathcal{O}}_X \left[\frac{X_1}{\xi}, \cdots, \frac{X_n}{\xi} \right]$$

PROOF. The graded ring of $\mathbf{B}_{\mathrm{dR}}^+[[X_1, \dots, X_n]]$ for the ideal $(\xi, X_1, \dots, X_n)$ is the polynomial ring $\widehat{\mathcal{O}}_X[\bar{\xi}, \bar{X}_1, \dots, \bar{X}_n]$, with every generator of degree one. After inverting $\bar{\xi}$, its degree-i part is

$$\bar{\xi}^i \widehat{\mathcal{O}}_X[\bar{X}_1/\bar{\xi}, \dots, \bar{X}_n/\bar{\xi}].$$

Proposition 5.34 identifies this with the desired graded sheaf. This description depends on the coordinates and on ξ; the graded sheaf itself does not. □

PROPOSITION 5.38. *Let $X = \mathrm{Spa}(R, R^+)$ be an affinoid adic space of finite type over $\mathrm{Spa}(k, \mathcal{O}_k)$, with an étale map $X \to \mathbf{T}^n$ that factors as a composition of rational embeddings and finite étale maps.*

(i) *Suppose K contains all p-power roots of unity. Then*

$$H^q(X_K, \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}) = \begin{cases} R\widehat{\otimes}_k K & q = 0 \\ 0 & q > 0 \end{cases}$$

(ii) *Let $\chi\colon \mathrm{Gal}(\bar{k}/k) \to \mathbf{Z}_p^\times$ be the cyclotomic character. Its logarithm defines a class*

$$\log \chi \in H^1_{\mathrm{cont}}(\mathrm{Gal}(\bar{k}/k), \mathbf{Q}_p).$$

Then

$$H^q(X, \mathrm{gr}^i \mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

vanishes unless $i = 0$ *and* $q = 0, 1$. *When* $i = 0$, *we have* $(\mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}})(X) = R$ *and* $H^1(X, \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}) = R \log \chi$.

Proof. There are two parts to the calculation: first the geometry of the toric cover, and then the Galois action on the coefficient field. We compute the cohomology of X_K using the cover $\tilde{X}_K \to X_K$, and the Cartan-Leray spectral sequence. The covering is a $\mathbf{Z}_p^n$ cover, so the spectral sequence is:

$$H^p_{\mathrm{cont}}(\mathbf{Z}_p^n, H^q(\tilde{X}_K, \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}})) \Rightarrow H^{p+q}(X_K, \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

Moreover, all of the fiber products $\tilde{X}_K \times_{X_K} \cdots \times_{X_K} \tilde{X}_K$ are affinoid perfectoid, and so the higher cohomology of all of the sheaves considered vanishes. The version of Corollary 5.31 for $\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ stays true, so we find that

$$H^q_{\mathrm{cont}}(\mathbf{Z}_p^n, \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\tilde{X}_K)) = H^q(X_K, \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

To compute this Galois cohomology group, we first write

$$\mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\tilde{X}_K) = \tilde{R}[V_1, \ldots, V_n]$$

where $\hat{\tilde{X}}_K = \mathrm{Spa}(\tilde{R}, \tilde{R}^+)$ and the V_i are given by $t^{-1} \log([T_i^\flat]/T_i)$, where $t = \log[\epsilon]$. If $\gamma_i \in \mathbf{Z}_p^n$ is the i-th basis vector, then the action of γ_i on V_j is given by

$$\gamma_i(V_j) = \begin{cases} V_j & i \neq j \\ V_i + 1 & i = j \end{cases}$$

Indeed, by definition γ_i acts trivially on $T_j^\flat$ if $i \neq j$, and by multiplication by ϵ if $i = j$. Granted this, we claim that the inclusion

$$(R\hat{\otimes}_k K)[V_1, \ldots, V_n] \subset \tilde{R}[V_1, \ldots, V_n]$$

induces an isomorphism on continuous $\mathbf{Z}_p^n$-cohomology. It suffices to check this on associated gradeds for the filtration given by the polynomial degree. But on associated gradeds, the action of $\mathbf{Z}_p^n$ on each V_i is trivial by the preceding observation, so it suffices to show that the inclusion $R\hat{\otimes}_k K \subset \tilde{R}$ induces an isomorphism on continuous $\mathbf{Z}_p^n$-cohomology.

Let us look first at what happens to the nonintegral monomials. The nice thing is that they contribute no cohomology once we invert p. Indeed, for a fractional exponent $a \notin \mathbf{Z}^n$, some generator γ_j acts on T^a by a nontrivial p-power root of unity ζ. The corresponding Koszul differential is multiplication by $\zeta - 1$, and contracting in that direction makes the Koszul complex acyclic. The inverses are uniformly bounded, since $|\zeta - 1| \geqslant |\zeta_p - 1|$ for all such roots. Thus this contraction extends from finite sums to the completed sum of fractional monomials. The integral monomials give exactly $R\hat{\otimes}_k K$.

It remains to account for the polynomial variables. For a fixed polynomial degree in the V_i, the same conclusion follows by the finite filtration by degree. Give the polynomial module its locally convex direct-limit topology over these finite-degree parts. Passing to all degrees is legitimate because continuous cochains from a compact group take values in a bounded-degree part of this module.

To proceed further, we relate R and $\tilde{R}$. The key result that allows us to do this is the following:

LEMMA 5.39. *The map*

$$R^+ \widehat{\otimes}_{\mathcal{O}_k\langle T_1^{\pm 1},\dots,T_n^{\pm 1}\rangle} \mathcal{O}_K\langle T_1^{\pm 1/p^\infty},\dots,T_n^{\pm 1/p^\infty}\rangle \to \tilde{R}^+$$

is injective, with cokernel annihilated by some power of p.

PROOF. Apply the approximation lemma for rational localizations and finite étale maps to the perfectoid torus and to $X \to \mathbf{T}^n$. It identifies the rational completed tensor product with $\widetilde{R}$ and shows that the image of the integral tensor product contains $p^N\widetilde{R}^+$ for some N. For injectivity, the integral perfectoid torus is a completed direct sum over the fractional monomials T^a, with $a \in \mathbf{Z}[1/p]^n/\mathbf{Z}^n$, over the ordinary integral torus after scalar extension. Tensoring and completing gives the same monomial decomposition with coefficients in $R^+\widehat{\otimes}_{\mathcal{O}_k}\mathcal{O}_K$. This module is p-torsion-free. Since the map becomes an isomorphism after inverting p, its kernel is p-power torsion and is therefore zero. This proves both assertions. □

Inverting p in the above lemma, we see that there is an isomorphism

$$R\widehat{\otimes}_{k\langle T_1^{\pm 1},\dots,T_n^{\pm 1}\rangle} K\langle T_1^{\pm 1/p^\infty},\dots,T_n^{\pm 1/p^\infty}\rangle \to \tilde{R}$$

Now we are reduced to computing $H^q_{\mathrm{cont}}(\mathbf{Z}_p^n, (R\widehat{\otimes}_k K)[V_1,\dots,V_n])$. We will do this inductively by showing that $H^q_{\mathrm{cont}}(\mathbf{Z}_p\gamma_i, (R\widehat{\otimes}_k K)[V_1,\dots,V_i]) = 0$ for $q > 0$ and equal to $(R\widehat{\otimes}_k K)[V_1,\dots,V_{i-1}]$ when $q = 0$. To do so, note that the cohomology is computed by the two-term complex

$$(R\widehat{\otimes}_k K)[V_1,\dots,V_i] \to (R\widehat{\otimes}_k K)[V_1,\dots,V_i]$$

where the map is $\gamma_i - 1$. But the kernel of $\gamma_i - 1$ is precisely $(R\widehat{\otimes}_k K)[V_1,\dots,V_{i-1}]$, and the cokernel of $\gamma_i - 1$ is trivial. For the latter assertion, use the basis of binomial polynomials: the difference operator satisfies $\Delta\binom{V_i}{m+1} = \binom{V_i}{m}$, and all the needed factorials are invertible in our coefficient ring. This proves (i). In fact, the same calculation gives $H^q(X_K, \mathrm{gr}^i\,\mathcal{O}\mathbf{B}_{\mathrm{dR}})$ for all $i \in \mathbf{Z}$, since all of these sheaves are isomorphic on $X_{\mathrm{pro\acute{e}t}}/X_K$ to $\mathrm{gr}^0\,\mathcal{O}\mathbf{B}_{\mathrm{dR}}$.

So far the calculation has been over the extended field K. To prove (ii), we have to come back down to our original base field. Let K be the completion of $k(\mu_{p^\infty})$, and let $\Gamma_k = \mathrm{Gal}(K/k)$. We will again use the Cartan-Leray spectral sequence for the cover $X_K \to X$, but for this we first need to compute $H^q(X_K^{m/X}, \mathrm{gr}^i\,\mathcal{O}\mathbf{B}_{\mathrm{dR}})$, where we set $X_K^{m/X} = X_K \times_X \cdots \times_X X_K$ (the m-fold product). Then we see that

$$H^q(X_K^{m/X}, \mathrm{gr}^i\,\mathcal{O}\mathbf{B}_{\mathrm{dR}}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Gamma_k^{m-1}, H^q(X_K, \mathrm{gr}^i\,\mathcal{O}\mathbf{B}_{\mathrm{dR}}))$$

In fact, using yet another Cartan-Leray spectral sequence for the cover $\tilde{X}_K \times_{X_K} X_K^{m/X}$, we see that

$$H^q(X, \mathrm{gr}^i\,\mathcal{O}\mathbf{B}_{\mathrm{dR}}) \cong H^q_{\mathrm{cont}}(\Gamma_k, R\widehat{\otimes}_k K(i))$$

At this point the geometry of the toric cover has disappeared from the problem, and we are left with a question about the continuous cohomology of the cyclotomic Galois group. We use Tate's cyclotomic trace estimate: the inclusion $k(i) \to K(i)$ induces an isomorphism on continuous Γ_k-cohomology, and the normalized-trace operators and their contracting estimates remain bounded after completed tensor product with an affinoid k-algebra. Thus the same assertion holds for $R(i) \to R\widehat{\otimes}_k K(i)$. This is the

arithmetic input from [Tat67]; it is separate from the geometric toric calculation. This gives

$$H^q_{\mathrm{cont}}(\Gamma_k, R\widehat{\otimes}_k K(i)) \cong H^q_{\mathrm{cont}}(\Gamma_k, R(i)) \cong R \otimes_{\mathbf{Q}_p} H^q_{\mathrm{cont}}(\Gamma_k, \mathbf{Q}_p(i))$$

It remains to compute the continuous cohomology groups on the right. Pass to an open procyclic subgroup of Γ_k. Its continuous cohomology is computed by the two-term operator $\gamma - 1$. On $\mathbf{Q}_p(i)$ this is the scalar $\chi(\gamma)^i - 1$, which is invertible for $i \neq 0$, and zero for $i = 0$. Taking invariants under the finite quotient is exact over $\mathbf{Q}_p$. Thus only degrees zero and one survive when $i = 0$, and the degree-one generator is the continuous homomorphism $\log \chi$. This gives exactly the groups in (ii). □

Since these calculations are local on $X_{\text{ét}}$, we can express their consequences in terms of direct images under ν. We will need the resulting formulas when working with vector bundles and their connections.

Corollary 5.40. *Let X be a smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Then $\nu_* \mathcal{O}\mathbf{B}_{\mathrm{dR}} = \mathcal{O}_{X_{\text{ét}}}$. Moreover, we have $\nu_* \widehat{\mathcal{O}}_X = \mathcal{O}_{X_{\text{ét}}}$, $\nu_* \widehat{\mathcal{O}}_X(n) = 0$ for $n \geqslant 1$, $R^1 \nu_* \widehat{\mathcal{O}}_X(1) \cong \Omega^1_{X_{\text{ét}}}$, and $R^1 \nu_* \widehat{\mathcal{O}}_X(n)$ vanishes for $n \geqslant 2$.*

Proof. These assertions can be checked on the étale basis of affinoids admitting the toric charts of Proposition 5.38. That proposition says

$$R^q \nu_* \operatorname{gr}^j \mathcal{O}\mathbf{B}_{\mathrm{dR}} = 0 \quad (j \neq 0), \qquad \nu_* \operatorname{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}} = \mathcal{O}_{X_{\text{ét}}}.$$

The complete filtration, its surjective quotient maps, and Lemma 5.13 then identify $\nu_* \mathcal{O}\mathbf{B}_{\mathrm{dR}}$ with $\mathcal{O}_{X_{\text{ét}}}$; a section has a finite lower filtration bound, and its only possible nonzero leading graded piece has degree zero.

The strict Poincaré lemma gives exact graded complexes

$$0 \to \widehat{\mathcal{O}}_X(n) \to \operatorname{gr}^n \mathcal{O}\mathbf{B}_{\mathrm{dR}} \to \operatorname{gr}^{n-1} \mathcal{O}\mathbf{B}_{\mathrm{dR}} \otimes \Omega^1_X \to \cdots.$$

For $n = 0$, the initial injection shows that $\nu_* \widehat{\mathcal{O}}_X$ lies in $\mathcal{O}_{X_{\text{ét}}}$, while the structural map gives the reverse inclusion. For $n \geqslant 1$, the first term after $\widehat{\mathcal{O}}_X(n)$ has zero direct image, so $\nu_* \widehat{\mathcal{O}}_X(n) = 0$. Computing degree one from the same resolution, the first two terms have zero direct images and first higher direct images if $n \geqslant 2$. Thus $R^1 \nu_* \widehat{\mathcal{O}}_X(n) = 0$ in this range. For $n = 1$, its degree-one term instead has direct image $\Omega^1_{X_{\text{ét}}}$, giving the asserted isomorphism. The first two terms of this calculation are exactly Faltings's extension, so the isomorphism is its connecting morphism. □

5. Filtered Modules and Integrable Connections

The Poincaré lemma has given us a useful way of thinking about $\mathbf{B}_{\mathrm{dR}}$: it consists of the horizontal sections of the structural period sheaf. We can extend this observation to modules with connection. Taking their horizontal sections will allow us to relate period local systems to the filtered vector bundles that appear on the de Rham side of the comparison.

Let k be a discretely valued, complete nonarchimedean extension of $\mathbf{Q}_p$, with perfect residue field κ. Let X be a smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$.

Definition 5.41. A $\mathbf{B}^+_{\mathrm{dR}}$-local system is a sheaf of $\mathbf{B}^+_{\mathrm{dR}}$-modules $\mathbf{M}$ that is pro-étale locally on X free of finite rank.

DEFINITION 5.42. An $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$-module with integrable connection is a sheaf of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$-modules $\mathcal{M}$ that is locally on $X_{\mathrm{pro\acute{e}t}}$ free of finite rank, together with an integrable connection $\nabla_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X$, which satisfies the Leibniz rule with respect to the usual derivation ∇ on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$. We use the natural completed topology and require the connection to be continuous. Locally, these are finite free complete modules; no assertion about arbitrary noncomplete modules is intended.

THEOREM 5.43. *The functor $\mathbf{M} \mapsto (\mathcal{M}, \nabla_{\mathcal{M}})$ given by $\mathcal{M} = \mathbf{M} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+, \nabla_{\mathcal{M}} = \mathrm{id} \otimes \nabla$ gives an equivalence between the categories of $\mathbf{B}_{\mathrm{dR}}^+$-local systems and the category of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$-modules with integrable connection. In fact, we may describe the inverse functor explicitly: it is given by $\mathbf{M} = \mathcal{M}^{\nabla_{\mathcal{M}}=0}$.*

PROOF. For a local system $\mathbf{M}$, the Poincaré lemma identifies the horizontal sections of $\mathbf{M} \otimes \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$ with $\mathbf{M}$. For the other direction, work on a toric perfectoid cover and write $S = B[[u_1, \dots, u_n]]$, where B is the ring of sections of $\mathbf{B}_{\mathrm{dR}}^+$. We must prove the formal assertion for a finite free complete S-module M with continuous integrable relative connection.

Let D_j be covariant differentiation in u_j. Integrability says $D_i D_j = D_j D_i$. For a multi-index I, write $D^I = \prod_j D_j^{I_j}$ and $I! = \prod_j I_j!$. The Taylor operator

$$h(m) = \sum_{I \in \mathbf{N}^n} \frac{(-u)^I}{I!} D^I(m)$$

is well defined in the u-adic topology, since only finitely many terms contribute modulo $(u)^N$. Termwise differentiation is legitimate, and the two contributions to $D_j h(m)$ cancel, so $h(m)$ is horizontal. The Leibniz rule similarly gives $h(u_j m) = 0$, while $h(m) \equiv m \pmod{(u)}$. Thus h induces a section $M/(u)M \to M^{\nabla=0}$ of reduction. A horizontal section with zero constant term is zero: compare the coefficients successively in total u-degree, using that all positive integers are invertible in B. Hence

$$M^{\nabla=0} \xrightarrow{\sim} M/(u)M.$$

Lifting a basis by h gives a horizontal basis of M, since its change-of-basis matrix is invertible modulo (u) and therefore invertible in the complete ring. Consequently $M = M^{\nabla=0} \otimes_B S$. The construction is compatible with restrictions, so horizontal sections form a $\mathbf{B}_{\mathrm{dR}}^+$-local system and both composites are naturally the identity. □

The period sheaves live on the pro-étale site, but the vector bundles that appear in de Rham cohomology are defined in the analytic topology. We should therefore check that changing the site has not introduced new vector bundles for the uncompleted structure sheaf. This is a descent statement: a bundle trivialized on a finer cover should descend together with its transition matrices.

LEMMA 5.44. *Let X_{an} be the site of open subsets of X. The following categories are equivalent.*

(i) *The category of $\mathcal{O}_{X_{\mathrm{an}}}$-modules $\mathcal{E}_{\mathrm{an}}$ that are locally on X_{an} free of finite rank.*
(ii) *The category of $\mathcal{O}_{X_{\acute{e}t}}$-modules $\mathcal{E}_{\acute{e}t}$ that are locally on $X_{\acute{e}t}$ free of finite rank.*
(iii) *The category of $\mathcal{O}_X$-modules $\mathcal{E}$ that are locally on $X_{pro\acute{e}t}$ free of finite rank.*

PROOF. There are natural morphisms of sites $\nu : X_{\text{proét}} \to X_{\text{ét}}$ and $\lambda : X_{\text{ét}} \to X_{\text{an}}$. We have seen that

$$\lambda_* \mathcal{O}_{X_{\text{ét}}} \cong \mathcal{O}_{X_{\text{an}}},$$
$$\nu_* \mathcal{O}_X \cong \mathcal{O}_{X_{\text{ét}}}.$$

These equalities give full faithfulness of the pullback functors. Indeed, after trivializing two bundles, a map between them is a matrix of sections of the structure sheaf. The equalities identify its entries on the two sites, and the resulting local maps agree on overlaps.

For essential surjectivity from the analytic to the étale site, use the local factorization of étale maps and work on an affinoid neighborhood. The descent question reduces to a finite étale surjection $Y = \operatorname{Spa}(B, B^+) \to X = \operatorname{Spa}(A, A^+)$. A bundle on Y corresponds to a finite projective B-module, and its identifications on $Y \times_X Y$ give a descent datum over $B \otimes_A B$. The map $A \to B$ is faithfully flat, so effective descent and descent of finite projectivity give a finite projective A-module [**The26**, Tags 023N and 058S]. The associated analytic bundle pulls back to the given one.

For the pro-étale step, suppose that $\mathcal{E}$ is trivial on a pro-étale cover U. After restricting to an étale neighborhood and refining the presentation, we may write $U = \varprojlim_i U_i$ with $U_0 = \operatorname{Spa}(A, A^+)$ and finite étale surjective transition maps. Put $U_i = \operatorname{Spa}(A_i, A_i^+)$. Here it matters that we are using the uncompleted structure sheaf:

$$\mathcal{O}_X(U) = A_\infty = \varinjlim_i A_i.$$

The map $A \to A_\infty$ is faithfully flat. Finite étale base change identifies the sections on double and triple pullbacks with the corresponding algebraic tensor products. Thus the transition matrices of $\mathcal{E}$ give an ordinary descent datum for $A \to A_\infty$, including its cocycle identity. The same descent theorem gives a finite projective A-module. On smaller analytic open subsets it is free, so its associated bundle recovers $\mathcal{E}$. This proves essential surjectivity and hence the equivalences. □

So far we have been looking at the underlying vector bundles and their connections. There is still an important piece of data that we have not put into the picture, namely the filtration. Recall that the condition relating the filtration and connection is Griffiths transversality.

DEFINITION 5.45. A filtered $\mathcal{O}_X$-module with integrable connection is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank on X, together with a locally finite, separated, exhaustive, decreasing filtration $\operatorname{Fil}^i \mathcal{E}$ by locally direct summands, and an integrable connection ∇ satisfying Griffiths transversality with respect to the filtration (i.e. $\nabla(\operatorname{Fil}^i \mathcal{E}) \subseteq \operatorname{Fil}^{i-1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$).

DEFINITION 5.46. Let $\mathcal{M}$ be an $\mathcal{O}\mathbf{B}^+_{\text{dR}}$-module with integrable connection, and $\mathcal{E}$ a filtered $\mathcal{O}_X$-module with integrable connection. We say that $\mathcal{M}$ and $\mathcal{E}$ are associated if there is an isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}\mathbf{B}^+_{\text{dR}}} \mathcal{O}\mathbf{B}_{\text{dR}} \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\text{dR}}$$

of sheaves on $X_{\text{proét}}$, compatible with filtrations and connections.

THEOREM 5.47. *Let $\mathcal{M}$ be an $\mathcal{O}\mathbf{B}^+_{\text{dR}}$-module with integrable connection, whose horizontal sections are given by $\mathbf{M}$. Suppose that $\mathcal{M}$ is associated to a filtered $\mathcal{O}_X$-module with integrable*

connection $\mathcal{E}$. Then we have

$$\mathbf{M} = \mathrm{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\mathrm{dR}})^{\nabla=0}$$

Moreover, we may also reconstruct $\mathcal{E}$ with its filtration by setting

$$\mathcal{E}_{\acute{e}t} \cong \nu_*(\mathbf{M} \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

Proof. Apply horizontal sections to the given comparison isomorphism. The Poincaré lemma identifies the horizontal sections on its left with $\mathbf{M} \otimes \mathbf{B}_{\mathrm{dR}}$. Compatibility with filtrations then gives

$$\mathrm{Fil}^0(\mathcal{E} \otimes \mathcal{O}\mathbf{B}_{\mathrm{dR}})^{\nabla=0} = \mathrm{Fil}^0(\mathbf{M} \otimes \mathbf{B}_{\mathrm{dR}}) = \mathbf{M}.$$

For reconstruction, the final corollary of the preceding section and the projection formula for the finite locally free bundle $\mathcal{E}$ give

$$\nu_*(\mathcal{E} \otimes \mathcal{O}\mathbf{B}_{\mathrm{dR}}) = \mathcal{E}_{\text{ét}}.$$

The associated isomorphism therefore gives the displayed formula for $\mathcal{E}_{\text{ét}}$. Its filtration is recovered by taking the direct image of the convolution filtration, and its connection by taking the direct image of the induced connection. Locally an adapted basis of $\mathcal{E}$ reduces the filtration assertion to $\nu_* \mathrm{Fil}^a \mathcal{O}\mathbf{B}_{\mathrm{dR}} = \mathcal{O}_{X_{\text{ét}}}$ for $a \leqslant 0$ and zero for $a > 0$, which follows from the graded calculation in the preceding section. Thus the reconstruction preserves all the indicated structure. □

Theorem 5.48. *For any filtered $\mathcal{O}_X$-module with integrable connection $\mathcal{E}$, the sheaf*

$$\mathbf{M} = \mathrm{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\mathrm{dR}})^{\nabla=0}$$

is a $\mathbf{B}^+_{\mathrm{dR}}$-local system such that $\mathcal{E}$ is associated to $\mathcal{M} := \mathbf{M} \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}$. In particular, the notion of being associated gives a fully faithful functor from the category of filtered $\mathcal{O}_X$-modules with integrable connection to the category of $\mathbf{B}^+_{\mathrm{dR}}$-local systems.

Proof. First ignore the filtration. The preceding equivalence gives

$$\mathbf{M}_0 = (\mathcal{E} \otimes \mathcal{O}\mathbf{B}^+_{\mathrm{dR}})^{\nabla=0}, \qquad \mathbf{M}_0 \otimes \mathcal{O}\mathbf{B}^+_{\mathrm{dR}} = \mathcal{E} \otimes \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}.$$

We now use induction on the length of the locally finite filtration, following [**Sch13b**, Theorem 7.6]. If the filtration has a single jump at a, the required local system is $t^{-a}\mathbf{M}_0$, and the assertion is immediate.

Otherwise, shift the filtration so that its largest nonzero step is $\mathrm{Fil}^0 \mathcal{E}$. Let $\mathrm{Fil}'^\bullet$ agree with $\mathrm{Fil}^\bullet$ except that $\mathrm{Fil}'^0 \mathcal{E} = 0$, and let $\mathbf{M}'$ be the local system given by induction. Set

$$C^q = \mathrm{Fil}^{-q}(\mathcal{E} \otimes \mathcal{O}\mathbf{B}_{\mathrm{dR}}) \otimes \Omega^q_X, \qquad C'^q = \mathrm{Fil}'^{-q}(\mathcal{E} \otimes \mathcal{O}\mathbf{B}_{\mathrm{dR}}) \otimes \Omega^q_X.$$

Griffiths transversality makes these complexes, and their quotient is

$$C^q/C'^q = \mathrm{Fil}^0 \mathcal{E} \otimes \mathrm{gr}^{-q} \mathcal{O}\mathbf{B}_{\mathrm{dR}} \otimes \Omega^q_X.$$

On this quotient the connection on $\mathcal{E}$ disappears: its contribution lands in the preceding filtration, while the remaining differential is $\mathrm{id} \otimes \nabla$. The strict Poincaré lemma shows that this quotient complex has only degree-zero cohomology, equal to $\mathrm{Fil}^0 \mathcal{E} \otimes \widehat{\mathcal{O}}_X$. By induction the same is true of $C'^\bullet$, with degree-zero cohomology $\mathbf{M}'$. The long exact sequence of the short exact sequence of complexes therefore gives

$$0 \longrightarrow \mathbf{M}' \longrightarrow \mathbf{M} \longrightarrow \mathrm{Fil}^0 \mathcal{E} \otimes \widehat{\mathcal{O}}_X \longrightarrow 0,$$

and proves exactness of the augmented complex for $\mathbf{M}$.

Tensor this sequence with $\mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}$ over $\mathbf{B}_{\mathrm{dR}}^+$. We use here the flatness of $\mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}$ over $\mathbf{B}_{\mathrm{dR}}^+$ in [Sch13b, Theorem 7.6]. Faithfulness can be checked directly on a toric cover: evaluating all u_j at zero retracts $\mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}$ onto $\mathbf{B}_{\mathrm{dR}}^+$, because the constant term of $t^{-m}(t, u_1, \dots, u_n)^m$ lies in $\mathbf{B}_{\mathrm{dR}}^+$. Thus the map is locally faithfully flat. Comparing with the corresponding exact sequence of $\mathrm{Fil}^0(\mathcal{E} \otimes \mathcal{O}\mathbf{B}_{\mathrm{dR}})$, the outside vertical arrows are isomorphisms: the first by induction and the last because $\mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}/t = \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}$. The middle arrow is therefore an isomorphism.

In an adapted local basis of the filtration, the target is finite free. Faithfully flat descent therefore shows that $\mathbf{M}$ is finite projective over $\mathbf{B}_{\mathrm{dR}}^+$ [The26, Tag 058S]. To obtain a local system, we still need a basis locally on the pro-étale site.

The reduction of $\mathbf{M}$ modulo t is finite projective over the perfectoid coefficient ring, so it becomes free on a rational cover. Lift a basis there. Completeness lets us correct an expression one power of t at a time, so these lifts generate. The resulting surjection from a finite free module splits by projectivity. Its kernel is a finite projective, complete module whose reduction modulo t is zero; successive division by t and separatedness force this kernel to vanish. Thus the lifted generators are a basis. The bounds on the filtration place $\mathbf{M}$ between two fixed powers of t times $\mathbf{M}_0$, ensuring the completeness used here. Thus $\mathbf{M}$ is a local system of finite rank.

Finally invert t. The isomorphism just constructed becomes the required filtered horizontal comparison with $\mathcal{E}$. This proves existence. Reconstruction in the preceding theorem recovers both objects and their morphisms, so the functor is fully faithful. □

It is useful to first forget the filtration and see what information about the connection we can recover from the period local system. For a pair $(\mathcal{E}, \nabla)$ equipped with the trivial filtration, the preceding theorem gives a $\mathbf{B}_{\mathrm{dR}}^+$-local system $\mathbf{M}_0 := (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+)^{\nabla=0}$ associated to it. It comes with an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+ \cong \mathbf{M}_0 \otimes_{\mathbf{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$$

which upon reducing modulo $\ker \theta$ gives an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X \cong \mathrm{gr}^0 \mathbf{M}_0$$

Keeping one more term of the filtration on $\mathbf{M}_0$ will allow us to recover the connection. Consider the short exact sequence

$$0 \to \mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(1) \to \mathbf{M}_0/\mathrm{Fil}^2 \mathbf{M}_0 \to \mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X \to 0$$

which via ν and the long exact sequence on cohomology induces a boundary map

$$\mathcal{E}_{\text{ét}} \to \mathcal{E}_{\text{ét}} \otimes R^1 \nu_*(\widehat{\mathcal{O}}_X(1))$$

Lemma 5.49. *Under the canonical isomorphism $R^1 \nu_*(\widehat{\mathcal{O}}_X(1)) \cong \Omega^1_{X_{\text{ét}}}$, the above boundary map is identified with -1 times the connection ∇ on $\mathcal{E}$.*

Proof. Pull the extension in the statement back along $\mathcal{E} \to \mathcal{E} \otimes \widehat{\mathcal{O}}_X$, and call the resulting middle sheaf $\mathcal{F}$. It has two maps into $\mathcal{E} \otimes (\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+/\mathrm{Fil}^2)$: one comes from its horizontal lift in $\mathbf{M}_0/\mathrm{Fil}^2$, and the other from its projection to $\mathcal{E}$. They agree modulo Fil^1. Their difference therefore maps to $\mathcal{E} \otimes \mathrm{gr}^1 \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+$.

The connection on the first lift is zero, while that on the second is ∇. Consequently this difference gives a morphism from the pulled-back extension to Faltings's extension

tensored with $\mathcal{E}$, whose map on right-hand quotients is $-\nabla$ and whose map on left-hand terms is the identity. Functoriality of connecting morphisms now identifies the boundary map with $-\nabla$, because the connecting morphism of Faltings's extension is the chosen isomorphism $\Omega^1_{X_{\text{ét}}} \cong R^1\nu_*\widehat{\mathcal{O}}_X(1)$. Locally the sign can also be read from the first-order horizontal lift $s - \sum_j u_j \nabla_{\partial/\partial T_j} s$, modulo Fil^2. □

Proposition 5.50. *The $\mathbf{B}^+_{\mathrm{dR}}$-local system $\mathbf{M}$ associated to $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$ is contained in $\mathbf{M}_0 \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}$, and satisfies*

$$(\mathbf{M} \cap \mathrm{Fil}^i \mathbf{M}_0)/(\mathbf{M} \cap \mathrm{Fil}^{i+1} \mathbf{M}_0) = \mathrm{Fil}^{-i} \mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(i) \subset \mathrm{gr}^i \mathbf{M}_0 \cong \mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_X(i)$$

for all $i \in \mathbf{Z}$. Conversely, there is a unique such $\mathbf{B}^+_{\mathrm{dR}}$-submodule in $\mathbf{M}_0 \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}$.

Proof. The inclusion follows by forgetting the filtration in the horizontal comparison. We prove the graded formula and uniqueness by the same induction on the filtration as in the preceding existence proof. For a single jump both assertions say that $\mathbf{M} = t^{-a}\mathbf{M}_0$.

Normalize the largest step to be zero, let $F = \mathrm{Fil}^0 \mathcal{E}$, and delete this step to obtain $\mathbf{M}'$. The construction above gives a diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{M}' & \longrightarrow & \mathbf{M} & \longrightarrow & F \otimes \widehat{\mathcal{O}}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & t\mathbf{M}_0 & \longrightarrow & \mathbf{M}_0 & \longrightarrow & \mathcal{E} \otimes \widehat{\mathcal{O}}_X & \longrightarrow & 0. \end{array}$$

The lower row is reduction modulo t. The upper row adds exactly the degree-zero piece F and leaves the lower steps unchanged. Induction gives the asserted intersection formula in every degree.

To justify uniqueness, including the extension data, work locally on the étale site and take extensions in the category of abelian sheaves on $X_{\text{proét}}$. For finite locally free $\mathcal{O}_X$-modules F_1, F_2, derived adjunction and the previously computed direct images give

$$\mathrm{Ext}^1(F_1, F_2 \otimes \widehat{\mathcal{O}}_X(k)) = 0 \quad (k \geqslant 2),$$

and for $k = 1$ identify this group with

$$\mathrm{Hom}(F_1, F_2 \otimes \Omega^1_X).$$

Indeed, $R\nu_*(F_2 \otimes \widehat{\mathcal{O}}_X(k))$ has zero degree-zero cohomology for $k \geqslant 1$ and zero degree-one cohomology for $k \geqslant 2$; for $k = 1$ its degree-one term is $F_{2,\text{ét}} \otimes \Omega^1_{X_{\text{ét}}}$. This also shows that the corresponding degree-zero Hom groups vanish.

Filter $\mathbf{M}'$ and $t\mathbf{M}_0$ by their intersections with powers of $t\mathbf{M}_0$. Their graded pieces have the form just considered, with $k \geqslant 1$. The long exact Ext sequences, first modulo t^N and then by completeness, give injections

$$\mathrm{Ext}^1(F, \mathbf{M}') \hookrightarrow \mathrm{Hom}(F, \mathrm{Fil}^{-1} \mathcal{E} \otimes \Omega^1_X), \qquad \mathrm{Ext}^1(F, t\mathbf{M}_0) \hookrightarrow \mathrm{Hom}(F, \mathcal{E} \otimes \Omega^1_X).$$

The first injection is compatible with the second under the inclusion $\mathrm{Fil}^{-1} \mathcal{E} \subset \mathcal{E}$, so the map between these Ext groups is injective. The inverse-limit step introduces no extra Hom term, since the positive-twist Hom groups vanish. Pull the lower row of the diagram back along $F \to \mathcal{E} \otimes \widehat{\mathcal{O}}_X$. Its class therefore comes from at most one extension by $\mathbf{M}'$. The same vanishing of Hom groups makes the embedding of that extension unique. A candidate $\mathbf{B}^+_{\mathrm{dR}}$-submodule is recovered from this pulled-back extension by

scalar extension and its prescribed graded quotients, so there is at most one candidate. The preceding construction supplies one. Finally, the boundary lemma identifies its extension class with the restriction of $-\nabla$; Griffiths transversality is exactly what allows that class to land in $\mathrm{Fil}^{-1}\,\mathcal{E}\otimes\Omega^1_X$. This completes the induction. □

We have seen how to pass from a filtered vector bundle with integrable connection to a period local system. The geometric acyclicity result is especially useful in computing the cohomology of this local system, as it brings us back to the de Rham complex of the original vector bundle.

THEOREM 5.51. *Let X be a proper smooth adic space over $\mathrm{Spa}(k,\mathcal{O}_k)$. Let $(\mathcal{E},\nabla,\mathrm{Fil}^\bullet)$ be a filtered module with integrable connection, with associated $\mathbf{B}^+_{\mathrm{dR}}$-local system $\mathbf{M}$. Let $\bar{k}$ be an algebraic closure of k, with completion $\hat{\bar{k}}$. Then there is a canonical isomorphism*

$$H^i(X_{\bar{k}},\mathbf{M})\otimes_{B^+_{\mathrm{dR}}} B_{\mathrm{dR}}\cong H^i_{\mathrm{dR}}(X,\mathcal{E})\otimes_k B_{\mathrm{dR}}$$

compatible with filtrations and Galois actions. Here, B_{dR} is Fontaine's field of p-adic periods, and B^+_{dR} is its valuation ring. Moreover, there is a Galois-equivariant isomorphism

$$H^i(X_{\bar{k}},\mathrm{gr}^0\,\mathbf{M})\cong\bigoplus_j H^{i-j,j}_{\mathrm{Hodge}}(X,\mathcal{E})\otimes_k\hat{\bar{k}}(-j)$$

where

$$H^{i-j,j}_{\mathrm{Hodge}}(X,\mathcal{E})=H^i(X,\mathrm{gr}^j(\mathrm{dR}(\mathcal{E})))$$

denotes the Hodge cohomology in bidegree $(i-j,j)$ of $\mathcal{E}$ using the de Rham complex $\mathrm{dR}(\mathcal{E})$ of $\mathcal{E}$ with its natural filtration.

PROOF. The de Rham complex of $\mathcal{E}$ is

$$0\to\mathcal{E}\to\mathcal{E}\otimes\Omega^1_X\to\cdots$$

where all of the maps are ∇. It is therefore filtered by subcomplexes of the form

$$0\to\mathrm{Fil}^m\,\mathcal{E}\to\mathrm{Fil}^{m-1}\,\mathcal{E}\otimes\Omega^1_X\to\cdots$$

where again the maps are all given by ∇. We may also replace $\mathbf{M}\otimes_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{B}_{\mathrm{dR}}$ by the quasi-isomorphic complex $\mathrm{dR}(\mathcal{E})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ with its natural filtration, given by

$$0\to\mathcal{E}\otimes\mathcal{O}\mathbf{B}_{\mathrm{dR}}\to\mathcal{E}\otimes\mathcal{O}\mathbf{B}_{\mathrm{dR}}\otimes\Omega^1_X\to\cdots$$

This gives rise to a natural map of filtered complexes $\mathrm{dR}(\mathcal{E})\to\mathrm{dR}(\mathcal{E})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbf{B}_{\mathrm{dR}}$, which induces a morphism

$$R\Gamma(X_{\bar{k}},\mathrm{dR}(\mathcal{E}))\otimes_{\bar{k}}B_{\mathrm{dR}}\to R\Gamma(X_{\bar{k}},\mathrm{dR}(\mathcal{E})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

We claim that the above map is a quasi-isomorphism in the filtered derived category. It suffices to check this on associated graded objects. By using the naive filtration on $\mathrm{dR}(\mathcal{E})$, we are reduced to showing the following:

LEMMA 5.52. *Let $\mathcal{A}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then for all $i\in\mathbf{Z}$, the natural map*

$$R\Gamma(X_{\bar{k}},\mathcal{A})\otimes_{\bar{k}}\mathrm{gr}^i\,B_{\mathrm{dR}}\to R\Gamma(X_{\bar{k}},\mathcal{A}\otimes_{\mathcal{O}_X}\mathrm{gr}^i\,\mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

is a quasi-isomorphism.

PROOF. By twisting, we may assume $i = 0$; in this case we get that $\mathrm{gr}^0 B_{\mathrm{dR}} = \widehat{\bar{k}}$. Then we are reduced to showing that

$$H^j(X_{\bar{k}}, \mathcal{A}) \otimes_{\bar{k}} \widehat{\bar{k}} \cong H^j(X_{\bar{k}}, \mathcal{A} \otimes_{\mathcal{O}_X} \mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}})$$

for all $j \geqslant 0$. To do this, we cover X by affinoid subsets on which $\mathcal{A}$ becomes free. Then the left-hand side becomes $H^j(X_{\mathrm{an}}, \mathcal{A}) \otimes_k \widehat{\bar{k}}$, and the right-hand side becomes $H^j(X_{\widehat{\bar{k}},\mathrm{an}}, \mathcal{A}_{\widehat{\bar{k}}})$ by Proposition 5.38 (i). On an acyclic affinoid cover, we can make the comparison on the complexes of sections. The proper coherent finiteness and completed base-change theorem then identifies these two groups, giving the desired quasi-isomorphism. □

Taking cohomology of this filtered quasi-isomorphism gives the first claimed comparison. Indeed, the Poincaré lemma identifies its right-hand complex with $R\Gamma(X_{\bar{k}}, \mathbf{M} \otimes \mathbf{B}_{\mathrm{dR}})$, and localization commutes with cohomology on the qcqs site, so its degree-i group is

$$H^i(X_{\bar{k}}, \mathbf{M}) \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}.$$

Proper coherent base change identifies the other side with

$$H^i_{\mathrm{dR}}(X, \mathcal{E}) \otimes_k B_{\mathrm{dR}}.$$

All maps were formed by pullback, the structural inclusions and the connection, so they preserve the filtrations and the Galois actions.

The reason we have been careful about filtrations is that we would also like to understand what happens on taking the associated graded. In degree zero this will give us the Hodge-Tate periods. The i-th cohomology of $\mathrm{gr}^0 \mathbf{M}$ is computed by

$$H^i(X_{\bar{k}}, \mathrm{gr}^0(\mathrm{dR}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\mathrm{dR}}))$$

and we have

$$\begin{aligned}
H^i(X_{\bar{k}}, \mathrm{gr}^0(\mathrm{dR}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbf{B}_{\mathrm{dR}})) &\cong H^i(X_{\bar{k}}, \bigoplus_j \mathrm{gr}^j \mathrm{dR}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathrm{gr}^{-j} \mathcal{O}\mathbf{B}_{\mathrm{dR}}) \\
&\cong \bigoplus_j H^i(X_{\bar{k}}, \mathrm{gr}^j \mathrm{dR}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathrm{gr}^{-j} \mathcal{O}\mathbf{B}_{\mathrm{dR}}) \\
&\cong \bigoplus_j H^i(X_{\bar{k}}, \mathrm{gr}^j \mathrm{dR}(\mathcal{E})) \otimes_{\bar{k}} \mathrm{gr}^{-j} B_{\mathrm{dR}} \\
&\cong \bigoplus_j H^{i-j,j}_{\mathrm{Hodge}}(X, \mathcal{E}) \otimes_k \widehat{\bar{k}}(-j)
\end{aligned}$$

For the third isomorphism we used the lemma just proved. In the last one we recognize the Hodge cohomology groups, with the appropriate Tate twists.

□

6. Comparison Theorems

We are quite close to the comparison we want. The last theorem gives us a way of comparing the cohomology of a period local system with de Rham cohomology. But we started with étale cohomology, and to get back to it we need to know that we can tensor with the period sheaf either before or after taking cohomology. To make this precise, let us first discuss lisse sheaves on the two sites.

DEFINITION 5.53. Let X be either a locally noetherian adic space or a locally noetherian scheme. A lisse $\mathbf{Z}_p$-sheaf $\mathbf{L}_\bullet$ on $X_{\text{ét}}$ is an inverse system of sheaves of $\mathbf{Z}/p^n\mathbf{Z}$-modules $\mathbf{L}_n$ on $X_{\text{ét}}$ such that each $\mathbf{L}_n$ is locally on $X_{\text{ét}}$ a constant sheaf associated to a finitely generated $\mathbf{Z}/p^n\mathbf{Z}$-module, and such that this inverse system is isomorphic in the pro-category to an inverse system for which $\mathbf{L}_{n+1}/p^n \cong \mathbf{L}_n$.

Let $\widehat{\mathbf{Z}}_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}$ be the sheaf on $X_{\text{proét}}$. Then a lisse $\widehat{\mathbf{Z}}_p$-sheaf is a sheaf $\mathbf{L}$ of $\widehat{\mathbf{Z}}_p$-modules on $X_{\text{proét}}$ such that locally on $X_{\text{proét}}$, $\mathbf{L}$ is isomorphic to $\widehat{\mathbf{Z}}_p \otimes_{\mathbf{Z}_p} M$, where M is a finitely generated $\mathbf{Z}_p$-module.

PROPOSITION 5.54. *Let X be a locally noetherian adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, and $\mathbf{L}_\bullet$ a lisse $\mathbf{Z}_p$-sheaf on $X_{ét}$. Then $\mathbf{L} := \widehat{\mathbf{L}_\bullet} := \varprojlim \nu^*\mathbf{L}_n$ is a lisse sheaf of $\widehat{\mathbf{Z}}_p$-modules on $X_{proét}$, inducing an equivalence of categories. Moreover, $R^j \varprojlim \nu^*\mathbf{L}_n = 0$ for $j > 0$. Moreover, letting $\widehat{\mathbf{Q}}_p := \widehat{\mathbf{Z}}_p[p^{-1}]$, we have that $\mathbf{L} \otimes_{\widehat{\mathbf{Z}}_p} \widehat{\mathbf{Q}}_p$ is a $\widehat{\mathbf{Q}}_p$-local system on $X_{proét}$.*

PROOF. Choose a representative of the inverse system with $\mathbf{L}_{n+1}/p^n = \mathbf{L}_n$. On a connected étale neighbourhood it is described by compatible finite continuous representations, hence by a continuous representation on a finitely generated $\mathbf{Z}_p$-module $M = \varprojlim M_n$. The successive finite covers trivializing the M_n form a countable tower of finite étale surjections. Their inverse limit is an allowed pro-étale cover on which

$$\varprojlim_n \nu^*\mathbf{L}_n = \widehat{\mathbf{Z}}_p \otimes_{\mathbf{Z}_p} M.$$

Thus the limit is lisse.

Conversely, reduction modulo p^n of a lisse $\widehat{\mathbf{Z}}_p$-sheaf is a finite locally constant sheaf. Its finitely many sections, transition maps and descent relations descend from a pro-étale presentation to a finite étale level. It therefore comes from a locally constant finite $\mathbf{Z}/p^n$-sheaf on $X_{\text{ét}}$. These descents are compatible with reduction, and a finite $\mathbf{Z}_p$-module is the inverse limit of its reductions. This proves essential surjectivity. The same descent applied to maps between finite reductions, followed by the inverse limit, proves full faithfulness.

For the derived assertion, let U be an arbitrary qcqs pro-étale object which is pro-finite étale over a connected affinoid U_0. Such objects form a basis after étale localization. Pull the universal connected finite étale tower $\widetilde{U}_0 \to U_0$ back to U. This gives an admissible cover

$$V = U \times_{U_0} \widetilde{U}_0 \longrightarrow U.$$

If $G = \pi_1(U_0, \bar{u})$ and U corresponds to a profinite G-set S, then V corresponds to $S \times G$ with diagonal action. This is a free profinite G-set; it need not be connected. The $K(\pi, 1)$ theorem on the finite étale stages and cohomological continuity identify finite-local-system cohomology on V with its cohomology on the profinite G-set site. Exactness of evaluation on free G-sets, proved earlier, makes this cohomology zero in positive degrees. Writing $S \times G \cong T \times G$ equivariantly, its degree-zero sections are the continuous maps $T \to M_n$. Such a map factors through a finite clopen partition of T, and its finitely many values lift along the surjection $M_{n+1} \to M_n$. Thus the section maps are surjective. The objects V obtained by these pullbacks form a covering basis, so Lemma 5.13 gives $R^j \varprojlim_n \nu^*\mathbf{L}_n = 0$ for $j > 0$. Finally $M[1/p]$ is a finite-dimensional $\mathbf{Q}_p$-vector space, so inverting p in the local description proves the last assertion. □

DEFINITION 5.55. Let k be a discretely valued complete nonarchimedean extension of $\mathbf{Q}_p$, with perfect residue field κ and ring of integers $\mathcal{O}_k$. Let X be a proper smooth adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. A lisse $\hat{\mathbf{Z}}_p$-sheaf $\mathbf{L}$ is said to be de Rham if the associated $\mathbf{B}_{\mathrm{dR}}^+$-local system $\mathbf{M} = \mathbf{L} \otimes_{\hat{\mathbf{Z}}_p} \mathbf{B}_{\mathrm{dR}}^+$ is associated to some filtered module with integrable connection $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$.

Putting the local systems and the filtered comparison together gives the following theorem. The first assertion is the comparison with $\mathbf{B}_{\mathrm{dR}}^+$ that we still need to prove. For de Rham local systems we then get the desired comparisons with de Rham and Hodge cohomology.

THEOREM 5.56. *Let k and $\mathcal{O}_k$ be as above, and fix an algebraic closure $\bar{k}$ of k. Let X be a smooth proper adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. Then for any lisse $\hat{\mathbf{Z}}_p$-sheaf $\mathbf{L}$ on $X_{pro\acute{e}t}$ with associated $\mathbf{B}_{\mathrm{dR}}^+$-local system $\mathbf{M} = \mathbf{L} \otimes_{\hat{\mathbf{Z}}_p} \mathbf{B}_{\mathrm{dR}}^+$, we have a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism*

$$H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}^+ \cong H^i(X_{\bar{k}}, \mathbf{M})$$

Moreover, if $\mathbf{L}$ is de Rham, with associated filtered module with integrable connection $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$, then the Hodge-de Rham spectral sequence

$$H_{\mathrm{Hodge}}^{i-j,j}(X, \mathcal{E}) \Rightarrow H_{\mathrm{dR}}^i(X, \mathcal{E})$$

degenerates, and there is a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism

$$H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}} \cong H_{\mathrm{dR}}^i(X, \mathcal{E}) \otimes_k B_{\mathrm{dR}}$$

that also preserves the natural filtrations on both sides. Finally, there is a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism

$$H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} \hat{\bar{k}} \cong \bigoplus_j H_{\mathrm{Hodge}}^{i-j,j}(X, \mathcal{E}) \otimes_k \hat{\bar{k}}(-j)$$

PROOF. We have already proved comparison with finite coefficients. Before passing to $\mathbf{Z}_p$, we must check that the inverse limit preserves the cohomology we want and that the resulting modules are finitely generated. The first question is controlled by the Mittag–Leffler condition; the second by a complete form of Nakayama's lemma. Let us spell out how these enter.

Put $C = \hat{\bar{k}}$, and let A_{inf} and B_{inf} denote the absolute period rings of C. Choose the normalized inverse system $\mathbf{L}_n = \mathbf{L}/p^n$. Theorem 5.24, followed by dévissage through powers of p, makes every $H^i(X_{\bar{k}}, \mathbf{L}_n)$ finite. These groups form Mittag–Leffler systems, since any inverse system of finite groups has stabilizing images. The preceding proposition therefore identifies $R\Gamma(X_{\bar{k}}, \mathbf{L})$ with the derived inverse limit of the finite-coefficient complexes. It is derived p-complete with bounded finite cohomology modulo p.

We still need to know that the cohomology is finitely generated over $\mathbf{Z}_p$, not just modulo p. Write the derived p-complete complex as D. The coefficient triangles show that $D \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{Z}/p^n$ has finite cohomology in a fixed bounded range. The cohomology-limit exact sequence and the Mittag–Leffler condition identify $H^i(D)$ with an inverse limit of finite p-groups, so it is a profinite $\mathbf{Z}_p$-module. The triangle for multiplication by p gives an injection $H^i(D)/p \hookrightarrow H^i(D \otimes_{\mathbf{Z}_p}^{\mathbf{L}} \mathbf{F}_p)$, whose target is finite.

We can now argue much as in Nakayama's lemma, using the profinite topology. Lift a finite set of generators modulo p. Their $\mathbf{Z}_p$-span is compact, hence closed, and the

quotient Q is profinite with $Q = pQ$. Every finite continuous quotient of Q is a finite p-group equal to its own multiple by p, and is therefore zero. Such quotients separate points, so $Q = 0$. This proves finite generation. If $\mathbf{L}$ has bounded torsion, its derived reductions also have a kernel term for multiplication by p^n; this is again a finite local system, and the same finiteness argument applies.

The integral conclusion in the proof of Theorem 5.24 gives the tilted primitive comparison

$$H^i(X_{\bar{k}}, \mathbf{L}_1) \otimes_{\mathbf{F}_p} \mathcal{O}^a_{C^\flat} \xrightarrow{\sim} H^i(X_{\bar{k}}, \mathbf{L}_1 \otimes \widehat{\mathcal{O}}^{+a}_{X^\flat}).$$

Since $A_{\mathrm{inf}}/p = \mathcal{O}_{C^\flat}$, this is the required A_{inf}-comparison for coefficients killed by p. For coefficients killed by p^n, use the short exact sequences for their powers of p and the five lemma. Both A_{inf} and the sheaf $\mathbf{A}_{\mathrm{inf}}$ are p-torsion-free and hence flat over $\mathbf{Z}_p$, so these short exact sequences remain exact after tensoring. We obtain

$$H^i(X_{\bar{k}}, \mathbf{L}_n) \otimes_{\mathbf{Z}_p} A^a_{\mathrm{inf}} \xrightarrow{\sim} H^i(X_{\bar{k}}, \mathbf{L}_n \otimes \mathbf{A}^a_{\mathrm{inf}}).$$

This is also an isomorphism on the corresponding derived complexes, since it holds in every degree.

On a perfectoid basis trivializing the finite coefficients, the right-hand sheaves are almost acyclic and their section maps are surjective. Lemma 5.13 applies. On the left, boundedness and finite generation give a bounded finite free representative for $R\Gamma(X_{\bar{k}}, \mathbf{L})$: over the discrete valuation ring $\mathbf{Z}_p$, every finite module has a finite free resolution of length at most one. We can therefore check the compatibility of tensor products with inverse limits term by term on finite free modules. On each term, p-adic completeness of A_{inf} identifies tensoring with A_{inf} with the inverse limit of the reductions; their transition maps are surjective, so the ordinary and derived inverse limits agree. For the bounded torsion part, any additional Tor terms in derived reduction form a pro-zero system, since their transition maps are multiplication by powers of p. Thus the normalized finite-coefficient system gives this same limit. Flatness of A_{inf} identifies its cohomology with $H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} A_{\mathrm{inf}}$. Thus

$$H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} A^a_{\mathrm{inf}} \xrightarrow{\sim} H^i(X_{\bar{k}}, \mathbf{L} \otimes \mathbf{A}^a_{\mathrm{inf}}).$$

Invert p to obtain the corresponding almost comparison over B_{inf}.

Choose ξ as in Lemma 5.28. Multiplication by ξ^m is injective on B_{inf} and on $\mathbf{B}_{\mathrm{inf}}$, and its cokernel is the quotient by $(\ker\theta)^m$. Apply the previous comparison to the two-term complexes for multiplication by ξ^m. Their cohomology exact sequences give

$$H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} (B_{\mathrm{inf}}/\xi^m) \xrightarrow{\sim} H^i(X_{\bar{k}}, \mathbf{L} \otimes (\mathbf{B}_{\mathrm{inf}}/\xi^m)).$$

This time the comparison is an honest isomorphism: the ideal defining the almost category contains $[\pi]$, whose image is a unit modulo ξ^m. There is no extra kernel term on the left, because $H^i(X_{\bar{k}}, \mathbf{L})[1/p]$ is a finite-dimensional $\mathbf{Q}_p$-vector space and multiplication by ξ^m remains injective after tensoring with it.

The section maps in the tower of these quotient sheaves are surjective on the perfectoid basis, and their higher cohomology there vanishes. Their global cohomology, by the displayed comparison, also has surjective transition maps. Lemma 5.13 and the cohomology-limit exact sequence therefore give

$$H^i(X_{\bar{k}}, \mathbf{L}) \otimes_{\mathbf{Z}_p} B^+_{\mathrm{dR}} \xrightarrow{\sim} H^i(X_{\bar{k}}, \mathbf{L} \otimes \mathbf{B}^+_{\mathrm{dR}}) = H^i(X_{\bar{k}}, \mathbf{M}).$$

Every map used is natural, so this is Galois equivariant and is compatible with the t-adic filtrations.

The groups on the right are finite free over B_{dR}^{+} in every degree. Apply the cohomology sequence of

$$0 \to \mathbf{M} \xrightarrow{t} \mathbf{M} \to \mathrm{gr}^0 \mathbf{M} \to 0.$$

Multiplication by t is injective in both degrees i and $i+1$. The long exact sequence shows that the cokernel of $H^i(X_{\bar{k}}, \mathbf{M})/t \to H^i(X_{\bar{k}}, \mathrm{gr}^0 \mathbf{M})$ is exactly the t-torsion in $H^{i+1}(X_{\bar{k}}, \mathbf{M})$. Here that term vanishes, so

$$H^i(X_{\bar{k}}, \mathrm{gr}^0 \mathbf{M}) = H^i(X_{\bar{k}}, \mathbf{M})/t.$$

It follows that the dimension of this group over C equals the rank of $H^i(X_{\bar{k}}, \mathbf{M})$ over B_{dR}^{+}. If $\mathbf{L}$ is de Rham, the two comparisons of Theorem 5.51 translate this equality into

$$\sum_j \dim_k H^{i-j,j}_{\mathrm{Hodge}}(X, \mathcal{E}) = \dim_k H^i_{\mathrm{dR}}(X, \mathcal{E}).$$

In a bounded spectral sequence of finite-dimensional vector spaces, every nonzero differential decreases the total dimension of its source and target diagonals on the next page. Equality on every diagonal therefore forces every differential to vanish. This proves degeneration. Inverting t in the first comparison and applying Theorem 5.51 gives the filtered de Rham comparison. Reducing modulo t and applying its Hodge comparison gives the final displayed isomorphism, with the indicated twists and Galois action. □

There is a second way of looking at the Hodge-Tate comparison when we have constant coefficients. Rather than starting with the de Rham complex, we can obtain a filtration directly from the pro-étale site. Suppose that k is a finite extension of $\mathbf{Q}_p$ and put $C = \widehat{\bar{k}}$. The higher direct images of $\widehat{\mathcal{O}}$ on the geometric space X_C give a spectral sequence

$$E_2^{a,b} = H^a(X, \Omega^b_{X/k}) \otimes_k C(-b) \Longrightarrow H^{a+b}_{\text{ét}}(X_{\bar{k}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C.$$

This is the Hodge-Tate spectral sequence. It is worth distinguishing it from the Hodge-de Rham spectral sequence: its abutment is geometric étale cohomology after extension to C, rather than de Rham cohomology over k. In particular, the two sequences start on different pages and give different filtrations. For the construction and its relation to the decomposition, see [**Sch13c**, Section 3.3]. Notice that in degree one we find the very two terms that appeared in Tate's theorem for abelian varieties at the beginning of the thesis.

CHAPTER 6

Applications

Recall that our motivation in Chapter 2 was to understand what distinguishes the p-adic representations that come from geometry. The comparison theorems give us a great deal of information about these representations. They also work in the other direction: we can use the Galois action on the cohomology of a space to learn something about its geometry. We will discuss some examples of both before turning to cohomology with integral coefficients.

Unless stated otherwise, K is a finite extension of $\mathbf{Q}_p$, $C = \mathbf{C}_K$, and X is a smooth proper rigid analytic space over K. We write X_C for its geometric base change and put

$$V^n = H^n_{\text{ét}}(X_C, \mathbf{Q}_p).$$

1. de Rham Representations

THEOREM 6.1. *The representation V^n is de Rham, and there is a natural identification*

$$\mathbf{D}_{\text{dR}}(V^n) \cong H^n_{\text{dR}}(X/K).$$

PROOF. The constant local system has associated filtered module with connection $(\mathcal{O}_X, d, \text{Fil}^\bullet)$, where $\text{Fil}^0 \mathcal{O}_X = \mathcal{O}_X$ and $\text{Fil}^1 \mathcal{O}_X = 0$. Theorem 5.56 therefore gives

$$B_{\text{dR}} \otimes_{\mathbf{Q}_p} V^n \cong B_{\text{dR}} \otimes_K H^n_{\text{dR}}(X/K).$$

On the right, G_K acts only on the first factor. Choosing a K-basis of the de Rham cohomology and using $B_{\text{dR}}^{G_K} = K$, we obtain the stated identification after taking invariants. Under this identification, the comparison map defining de Rham admissibility is exactly the displayed isomorphism. □

This gives us a large class of de Rham representations to start with. It would also be useful to know which of the usual constructions in representation theory give us new de Rham representations from these.

PROPOSITION 6.2. *Both de Rham representations and Hodge-Tate representations are closed under finite direct sums, tensor products, duals, subrepresentations, quotients, and Tate twists. If*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is exact and V belongs to either class, applying the corresponding functor $\mathbf{D}_{\text{dR}}$ or $\mathbf{D}_{\text{HT}}$ gives a short exact sequence.

PROOF. The same argument works for $B = B_{\text{dR}}$ and $B = B_{\text{HT}}$. Put $D_B(T) = (B \otimes_{\mathbf{Q}_p} T)^{G_K}$ and $d_B(T) = \dim_K D_B(T)$. Tensoring with B is exact, and taking invariants is left exact, so

$$d_B(V) \leqslant d_B(U) + d_B(W) \leqslant \dim_{\mathbf{Q}_p} U + \dim_{\mathbf{Q}_p} W = \dim_{\mathbf{Q}_p} V.$$

If V is admissible, the first and last quantities agree. Both inequalities must therefore be equalities. Lemma 2.14 shows that U and W are admissible, and equality of dimensions shows that $D_B(V) \to D_B(W)$ is surjective.

If V and W are admissible, choose invariant bases of $B \otimes V$ and $B \otimes W$. Their tensor products form an invariant basis of $B \otimes (V \otimes W)$, and the dual basis is invariant in $B \otimes V^\vee$. These bases prove the tensor and dual assertions. Direct sums work in the same way. Finally, if e_r is a basis of $\mathbf{Q}_p(r)$, then $t^{-r} \otimes e_r$ is an invariant basis after tensoring with either period ring. This proves the assertion about twists. □

In particular, every Galois-stable subquotient of V^n, and every tensor construction made from such subquotients and their duals, is de Rham. Symmetric and exterior powers are included: in characteristic zero they are images of the usual idempotents acting on tensor powers.

Recall that the original reason we were interested in de Rham representations was Berger's monodromy theorem. Applying it here, we see that all the cohomology representations of a smooth proper rigid space become semistable. In fact, we can make this happen over a single finite extension.

COROLLARY 6.3. *There is a finite extension L/K over which all the representations $H^n_{\text{ét}}(X_C, \mathbf{Q}_p)$ are semistable in the p-adic sense.*

PROOF. Theorem 6.1 and Theorem 2.28 give such an extension for each n. Only finitely many degrees occur. Taking the compositum of these finitely many extensions gives a single finite extension, since semistability is preserved by further finite extension. □

2. Recovering Hodge Numbers

So far, we have started with a geometric object and used it to say something about a Galois representation. We can also go in the other direction. The Hodge-Tate decomposition, together with Tate's calculation of invariants, allows us to recover the individual Hodge numbers of X from V^n.

PROPOSITION 6.4. *For every integer j, there is a natural isomorphism*

$$(C(j) \otimes_{\mathbf{Q}_p} V^n)^{G_K} \cong H^{n-j}(X, \Omega^j_{X/K}),$$

with the right side understood to be zero outside its usual range.

PROOF. Tensor the Hodge-Tate decomposition of Theorem 5.56 with $C(j)$ to obtain

$$C(j) \otimes_{\mathbf{Q}_p} V^n \cong \bigoplus_a H^{n-a}(X, \Omega^a_{X/K}) \otimes_K C(j-a).$$

The cohomology groups on the right have trivial Galois action. Theorem 2.12 kills the invariant part of every summand except $a = j$, and for that summand $C^{G_K} = K$. Taking invariants proves the claim. □

To translate this into Hodge-Tate numbers, recall our sign convention from Chapter 2: $h_i(V) = \dim_K (C(-i) \otimes V)^{G_K}$. Thus

$$h_{-j}(V^n) = \dim_K H^{n-j}(X, \Omega^j_{X/K}).$$

If X has pure dimension d, the only possible indices j lie between $\max(0, n-d)$ and $\min(n, d)$, since differential forms and coherent cohomology vanish outside these ranges.

COROLLARY 6.5. *Every Galois-stable subquotient of V^n is Hodge-Tate, and its Hodge-Tate numbers can be nonzero only at the indices occurring for V^n.*

PROOF. Apply Proposition 6.2 with $B = B_{\mathrm{HT}}$. All the maps in the resulting exact sequences preserve the grading. Taking a fixed graded piece is exact, so a piece that vanishes for V^n also vanishes for a subrepresentation and for a quotient. Applying this twice handles a subquotient. □

For example, let X be a smooth proper geometrically connected curve of genus g. The decomposition in degree one is

$$C \otimes_{\mathbf{Q}_p} H^1_{\text{ét}}(X_C, \mathbf{Q}_p) \cong \big(H^1(X, \mathcal{O}_X) \otimes_K C\big) \oplus \big(H^0(X, \Omega^1_{X/K}) \otimes_K C(-1)\big).$$

Both coherent cohomology groups have dimension g, by the definition of genus and Serre duality. Thus the representation has dimension $2g$, with Hodge-Tate numbers $h_0 = h_{-1} = g$. For an elliptic curve, the rational Tate module is dual to this cohomology representation, which explains the numbers $h_0 = h_1 = 1$ appearing in Chapter 2.

More generally, an isomorphism between the geometric p-adic cohomology representations of two smooth proper spaces in degree n forces equality of all their Hodge numbers in that degree. This follows directly from Proposition 6.4: these dimensions can be read from the representation itself.

3. Betti Numbers and Degeneration

Even if we forget the individual Hodge numbers and just add them all up in a given degree, we obtain a useful equality of dimensions. Write $b_n = \dim_{\mathbf{Q}_p} V^n$. Taking dimensions in the Hodge-Tate comparison gives

$$b_n = \sum_j \dim_K H^{n-j}(X, \Omega^j_{X/K}).$$

Taking dimensions in the de Rham comparison gives $b_n = \dim_K H^n_{\mathrm{dR}}(X/K)$. These two equalities also explain the degeneration assertion in Theorem 5.56.

PROPOSITION 6.6. *The Hodge-de Rham spectral sequence*

$$E_1^{a,b} = H^b(X, \Omega^a_{X/K}) \Longrightarrow H^{a+b}_{\mathrm{dR}}(X/K)$$

degenerates at E_1. Moreover,

$$\sum_n (-1)^n b_n = \sum_a (-1)^a \chi(X, \Omega^a_{X/K}),$$

where $\chi(X, \mathcal{F}) = \sum_b (-1)^b \dim_K H^b(X, \mathcal{F})$.

PROOF. All the vector spaces involved are finite-dimensional, and only finitely many are nonzero. The two dimension equalities above imply

$$\sum_{a,b} \dim_K E_1^{a,b} = \sum_n \dim_K H^n_{\mathrm{dR}}(X/K) = \sum_{a,b} \dim_K E_\infty^{a,b}.$$

Passing from a page to the next takes cohomology. A nonzero differential decreases the total dimension by twice its rank. Since the initial and final total dimensions agree, every differential must vanish. This proves degeneration. For the Euler characteristic identity, multiply the formula for b_n by $(-1)^n$, sum over n, and put $b = n - a$. □

Notice that the Hodge filtration itself remains part of the data. Degeneration identifies its graded pieces; it does not by itself choose a splitting of the filtered K-vector space $H^n_{\mathrm{dR}}(X/K)$.

4. Galois-Invariant Classes and Algebraic Cycles

The calculation of invariants above is useful for more than counting dimensions. It also restricts which summands in the Hodge-Tate decomposition can contain a Galois-invariant cohomology class after a Tate twist. This is particularly useful in studying classes coming from algebraic cycles.

PROPOSITION 6.7. *For every $r \geqslant 0$, there is a natural injective map*

$$K \otimes_{\mathbf{Q}_p} H^{2r}_{\acute{e}t}(X_C, \mathbf{Q}_p(r))^{G_K} \longrightarrow H^r(X, \Omega^r_{X/K}).$$

In particular,

$$\dim_{\mathbf{Q}_p} H^{2r}_{\acute{e}t}(X_C, \mathbf{Q}_p(r))^{G_K} \leqslant \dim_K H^r(X, \Omega^r_{X/K}).$$

PROOF. Let $W = H^{2r}_{\text{ét}}(X_C, \mathbf{Q}_p(r))$ and choose a $\mathbf{Q}_p$-basis of W^{G_K}. It is part of a $\mathbf{Q}_p$-basis of W, so it remains independent after extending scalars to C. Consequently the map

$$K \otimes_{\mathbf{Q}_p} W^{G_K} \longrightarrow (C \otimes_{\mathbf{Q}_p} W)^{G_K}$$

is injective. Proposition 6.4, with $n = 2r$ and $j = r$, identifies the target with $H^r(X, \Omega^r_{X/K})$. This proves both assertions. □

Suppose now that X is the analytification of a smooth projective variety over K. The usual étale cycle-class construction sends a codimension-r algebraic cycle defined over K to a Galois-invariant class in $H^{2r}_{\text{ét}}(X_C, \mathbf{Q}_p(r))$. Proposition 6.7 therefore places its Hodge-Tate image in the (r, r)-part. This is a necessary condition on a cycle class. A converse asserting that suitable Galois-invariant classes come from cycles is a much deeper question.

For instance, any collection of such cycles whose étale classes are linearly independent over $\mathbf{Q}_p$ has size at most $\dim_K H^r(X, \Omega^r_{X/K})$. We obtain this bound without choosing equations for the cycles: the Galois action and the comparison theorem already impose it.

5. Integral p-adic Hodge Theory

Throughout the thesis we have largely worked with rational cohomology. In passing from $\mathbf{Z}_p$-coefficients to $\mathbf{Q}_p$-coefficients, however, we lose any information about p-torsion. The integral p-adic Hodge theory of Bhatt-Morrow-Scholze [**BMS18**] allows us to retain some of this information.

For this purpose, it is natural to study both the generic and special fibers of a smooth proper object over $\mathcal{O}_{\mathbf{C}_K}$. In particular, another cohomology theory comes into play, namely *crystalline* cohomology of the special fiber X_k. Just as étale cohomology has a Galois action and de Rham cohomology has a filtration, crystalline cohomology has a *Frobenius* action. There is also an associated notion of a crystalline representation and a crystalline period ring B_{crys}. We refer to [**FO**] for the details of these constructions; here we will mainly be interested in the comparison results.

THEOREM 6.8 (Bhatt-Morrow-Scholze, [BMS18, Theorem 1.1]). *Let X be a smooth proper formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in some complete discretely valued nonarchimedean extension K of $\mathbf{Q}_p$ with perfect residue field k. Let $\mathbf{C}_K$ be a completed algebraic closure of K, and let $X_{\mathbf{C}_K}$ denote the rigid analytic generic fiber of X after base change to $\mathbf{C}_K$.*

(i) *There is a comparison isomorphism*

$$H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} B_{\text{crys}} \cong H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} B_{\text{crys}}$$

that is compatible with filtrations, Galois actions and Frobenius actions.

(ii) *For all $n \geqslant 0$, we have the following inequality:*

$$\text{length}_{W(k)}(H^i_{\text{crys}}(X_k/W(k))_{\text{tor}}/p^n) \geqslant \text{length}_{\mathbf{Z}_p}(H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{Z}_p)_{\text{tor}}/p^n)$$

(iii) *Suppose that both $H^i_{\text{crys}}(X_k/W(k))$ and $H^{i+1}_{\text{crys}}(X_k/W(k))$ are p-torsion-free. Then the crystalline cohomology group $H^i_{\text{crys}}(X_k/W(k))$ with its Frobenius action can be recovered from the étale cohomology $H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{Z}_p)$ with its Galois action.*

The key idea of Bhatt-Morrow-Scholze in the proof of the above theorem was to define a new cohomology theory, so-called A_{inf}-cohomology, that relates these different cohomology theories by suitable base changes. This was later generalized to *prismatic cohomology* by Bhatt-Scholze [BS22].

We will not construct A_{inf}-cohomology here. One particularly nice feature of the theory, though, is that we can recover the comparison that we have spent much of the thesis studying. More precisely, Bhatt-Morrow-Scholze obtain a comparison over B^+_{dR} which, in the situation below, gives back Theorem 5.56.

THEOREM 6.9 (Bhatt-Morrow-Scholze, [BMS18, Theorem 1.7]). *Let X be a smooth proper adic space over $\mathbf{C}_K$. Then there are cohomology groups $H^i_{\text{crys}}(X/B^+_{\text{dR}})$, which come with canonical isomorphisms*

$$H^i_{\text{crys}}(X/B^+_{\text{dR}}) \otimes_{B^+_{\text{dR}}} B_{\text{dR}} \cong H^i_{\text{ét}}(X, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} B_{\text{dR}}$$

If $X = X_0 \otimes_K \mathbf{C}_K$ arises by base change from a smooth proper adic space X_0 over some complete, discretely valued extension K of $\mathbf{Q}_p$ with perfect residue field k, then this isomorphism recovers the étale-de Rham comparison isomorphism of Theorem 5.56, via the identification

$$H^i_{\text{crys}}(X/B^+_{\text{dR}}) = H^i_{\text{dR}}(X_0) \otimes_K B^+_{\text{dR}}$$

In particular, the integral comparison allows us to bound the torsion in étale cohomology. This also tells us something about the dimensions of the cohomology groups with finite coefficients.

COROLLARY 6.10. *In the situation of Theorem 6.8, if $H^i_{\text{crys}}(X_k/W(k))$ is p-torsion-free, then $H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{Z}_p)$ is p-torsion-free as well.*

PROOF. Write T for the torsion submodule of the latter group. Taking $n = 1$ in the length inequality gives $\text{length}_{\mathbf{Z}_p}(T/pT) = 0$, so $T = pT$. The group T is a finite p-primary torsion module, and hence $p^N T = 0$ for some N. Iterating $T = pT$ gives $T = p^N T = 0$. □

COROLLARY 6.11. *In the situation of Theorem 6.8, one has*

$$\dim_{\mathbf{F}_p} H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{F}_p) \leqslant \dim_k H^i_{\text{dR}}(X_k/k).$$

PROOF. Reducing a cohomology group modulo p is slightly different from taking cohomology with mod-p coefficients. The difference comes from p-torsion in the next degree, as the long exact sequence of coefficients makes explicit. Put $E^j = H^j_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{Z}_p)$ and $C^j = H^j_{\text{crys}}(X_k/W(k))$. The coefficient sequence for multiplication by p gives

$$0 \longrightarrow E^i/pE^i \longrightarrow H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{F}_p) \longrightarrow E^{i+1}[p] \longrightarrow 0.$$

On the crystalline side we use the standard derived base-change identification

$$R\Gamma_{\text{crys}}(X_k/W(k)) \otimes^{\mathbf{L}}_{W(k)} k \simeq R\Gamma_{\text{dR}}(X_k/k),$$

which gives the corresponding exact sequence with C^i/pC^i and $C^{i+1}[p]$.

The structure theorem for finite modules over a discrete valuation ring says that a torsion module and its subgroup killed by p have the same number of cyclic summands. If $a_j = \dim_{\mathbf{F}_p}(E^j_{\text{tor}}/p)$ and $c_j = \dim_k(C^j_{\text{tor}}/p)$, the two exact sequences therefore give

$$\dim_{\mathbf{F}_p} H^i_{\text{ét}}(X_{\mathbf{C}_K}, \mathbf{F}_p) = b_i + a_i + a_{i+1},$$
$$\dim_k H^i_{\text{dR}}(X_k/k) = b_i + c_i + c_{i+1}.$$

Here the ranks on the two sides are the same number b_i by the crystalline comparison isomorphism. The length inequality in degrees i and $i+1$, with $n = 1$, gives $a_i \leqslant c_i$ and $a_{i+1} \leqslant c_{i+1}$, proving the claim. □

Notice that when we reduce modulo p, it is not enough to know the torsion in just the cohomological degree we are interested in. The torsion in the next degree enters as well. For instance, if crystalline cohomology is torsion-free in degrees i and $i+1$, both dimensions in Corollary 6.11 equal the rational Betti number b_i. None of this information about torsion was visible in the rational comparison. It is rather remarkable that one can obtain results of this kind by comparing the different cohomology theories integrally.

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