

GAGA FOR PROPER SCHEMES OVER THE FARGUES–FONTAINE CURVE

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ABSTRACT. We prove GAGA for proper schemes over the Fargues–Fontaine curve. Analytification gives an equivalence on pseudocoherent complexes and commutes with proper direct images. This yields the coherent, bounded-derived, perfect, and vector-bundle comparisons for Huber analytification.

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1. INTRODUCTION

1.1. GAGA and the relative problem. In this paper, we prove GAGA for every scheme proper over the Fargues–Fontaine curve. Coherent sheaves on the schematic curve already agree with coherent sheaves on its adic counterpart, and their cohomology agrees as well [KL19, Theorems 4.7.4–4.7.5]. We extend this comparison to proper schemes over the curve, answering the relative question discussed in [Ked23, Ked24].

We will use the boundary-localization argument of [CS26, Theorem 13.10] in its nonarchimedean form [LH26, A.13–A.18]. The idea is to recover a complex on each affine open from its direct image to the base. Proper finiteness makes this direct image accessible, and full faithfulness recovers the action of the coordinate algebra. The affinoid comparison and descent results are from [And21].

For classical GAGA, see [Ser56]. Proper GAGA over strongly noetherian Tate affinoids is proved in [FK18, Theorem II.9.5.1]; see also [Zav25, Lemmas 6.7 and 6.9]. Other approaches to GAGA appear in [Hal22, Wan26].

1.2. The Fargues–Fontaine curve. Let E be a nonarchimedean local field with residue field $\mathbf{F}_q$, and let F be a complete algebraically closed perfectoid field of characteristic p containing $\mathbf{F}_q$. Write $X = X_{F,E}^{\text{sch}}$ and $\mathcal{X} = X_{F,E}^{\text{ad}}$, and let $\lambda : \mathcal{X} \rightarrow X$ be the comparison map [FF18] [FS24, Section II.1]. The scheme X is a noetherian regular separated integral curve, and the analytic period rings used below are strongly noetherian [KL19, Theorem 4.7.3]. We fix F and X throughout.

For $\mathcal{F} \in \text{Coh}(X)$, our starting point is

$$\begin{aligned} \lambda^* : \text{Coh}(X) &\xrightarrow{\sim} \text{Coh}(\mathcal{X}), \\ H^i(X, \mathcal{F}) &\xrightarrow{\sim} H^i(\mathcal{X}, \lambda^* \mathcal{F}) \end{aligned}$$

[KL19, Theorems 4.7.4–4.7.5]. A *standard affine open* of X means one from its Proj presentation. All relative analytifications are taken over $\mathcal{X}$. Since X is noetherian, every proper X -scheme is of finite presentation [Stacks, Tags 01W1 and 01TX]; smoothness and projectivity are not among our hypotheses.

1.3. Coefficient categories and realizations. We use cohomological indexing and regard derived categories as stable ∞ -categories. Write $D_{\text{qc}}(T)$ for quasicoherent complexes and $D_{\text{pc}}(T)$ for pseudocoherent complexes. For noetherian T , we have $D_{\text{pc}}(T) = D_{\text{coh}}^-(T)$. The symbol Map denotes the mapping spectrum in a stable category, and the mapping space when the category is not stable.

For an analytic stack $\mathcal{Z}$, write $\mathcal{C}_{\square}(\mathcal{Z})$ for its presentable stable category of solid quasicoherent complexes. Over a complete Tate Huber pair (R, R^+) , we use

$$\begin{aligned} D_{\square}(R, R^+) &= D((R, R^+)^{\square}), \\ \text{Cond}_R : D(R) &\longrightarrow D_{\square}(R, R^+). \end{aligned}$$

Here D denotes complete modules and Cond_R is condensification. For an ordinary ring B , let $D_{\text{cond}}(B)$ be the derived category of condensed B_{disc} -modules; a subscript conn denotes its connective part.

For a separated finitely presented X -scheme T , write $T_{\mathcal{X}}^{\text{Alg}}$, $T_{\mathcal{X}}^{\text{An}}$, and $T_{\mathcal{X}}^{\text{ad}}$ for its algebraic realization as an analytic stack, the analytic locus of that realization, and Huber analytification. Over $V = \text{Spa}(R, R^+) \subset \lambda^{-1}(W)$, with $W \subset X$ affine, we construct these from $T_R = T \times_X \text{Spec } R$ and glue as V varies. Properness gives $T_{\mathcal{X}}^{\text{An}} \simeq T_{\mathcal{X}}^{\text{Alg}}$ [LH26, A.9–A.13]. Huber’s mapping property defines $T_{\mathcal{X}}^{\text{ad}}$ [Hub94, Proposition 3.8].

Let $D_{\text{pc}}^{\square}$ be the full subcategory of complexes that are nuclear and pseudocompact on the charts in [LH26, A.15]. On a base affinoid, this is $\text{Cond}_R(D_{\text{pc}}(R))$. We compute its cohomology and truncations using these ordinary affinoid representatives. For proper relative spaces, Section 6 identifies this convention with the coherent t -structure transported through GAGA. The ambient category $\mathcal{C}_{\square}$ is used for tensor products, descent, and colimits [CS26, Section 12].

Appendix A recalls the definitions, and Appendix B proves the categorical and analytic lemmas used below.

1.4. Main results. Let $f : Y \rightarrow X$ be proper. We prove that

$$\alpha_Y : D_{\text{pc}}(Y) \xrightarrow{\sim} D_{\text{pc}}^{\square}(Y_{\mathcal{X}}^{\text{An}})$$

is an equivalence and that the canonical map

$$\alpha_X(\mathbf{R}f_*M) \longrightarrow \mathbf{R}f_*^{\text{An}}\alpha_Y(M)$$

is an equivalence for $M \in D_{\text{pc}}(Y)$ (Theorem 5.5). Passing to Huber analytification gives the comparisons for coherent sheaves, bounded coherent complexes, perfect complexes, and vector bundles, together with cohomology and proper direct images (Theorem 6.14).

The main calculation is the following. Let $C \in D_{\text{pc}}^{\square}(Y_{\mathcal{X}}^{\text{An}})$ and choose an affine open $j : U \hookrightarrow Y$ for which $q = f \circ j$ is affine. Blow up the complement to obtain $p : \tilde{Y} \rightarrow Y$, an isomorphism over U , with effective Cartier boundary D . Put $\tilde{f} = f \circ p$ and $\tilde{C} = (p^{\text{An}})^*C$. Theorem 5.2 gives

$$\mathbf{R}q_*^{\text{Alg}}j^{\text{Alg},*}C \simeq \text{colim}_{n \geq 0} \mathbf{R}\tilde{f}_*^{\text{An}}(\tilde{C} \otimes \mathcal{O}_{\tilde{Y}}(nD)).$$

The transition maps are induced by the section of $\mathcal{O}(D)$. Proper finiteness makes every term on the right pseudocoherent on $\mathcal{X}$. We lift this sequence to a diagram $M_\bullet(C)$ in $D_{\text{pc}}(X)$ and obtain

$$\mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*} C \simeq \alpha_X \left(\text{colim}_n M_n(C) \right).$$

The colimit, with this comparison, is natural in C and independent of the boundary blowup. It gives the underlying coefficients of the desired complex on U . We then lift the action of $q_* \mathcal{O}_U$ and check pseudocoherence by faithfully flat descent.

For this construction, we first prove that

$$\alpha_X : D_{\text{qc}}(X) \longrightarrow \mathcal{C}_\square(\mathcal{X})$$

is fully faithful, sends perfect complexes to compact objects, and has image closed under colimits (Theorem 2.6). Proposition 2.10 identifies its pseudocoherent part. Applying α_X to module objects gives

$$\alpha_T : D_{\text{qc}}(T) \longrightarrow \mathcal{C}_\square(T_{\mathcal{X}}^{\text{Alg}})$$

fully faithfully for every separated finitely presented X -scheme T (Proposition 3.7). Proposition 4.10 then compares the analytic realization and its coefficient categories with Huber analytification over a strongly noetherian Tate base.

1.5. Relation to the work Clausen–Scholze. Clausen–Scholze use their theory of condensed mathematics to prove a much more general GAGA theorem than Serre’s classical statement; we roughly follow their proof outline in this article. They first identify the large algebraic and analytic coefficient categories using properness. They then recover pseudocoherent objects by restricting to an affine open and writing its direct image as a colimit of proper direct images of boundary twists. Their theorem applies to proper derived schemes over a complex affinoid algebra [CS26, Theorems 7.2, 12.18, and 13.10]. We use the Tate-base version of this argument from [LH26, A.13 and A.16–A.18].

Over $(X, \mathcal{X})$, we need to lift the whole sequence to the schematic curve. The enhanced base comparison supplies this lift and allows us to take its colimit. The affine module description recovers the algebra action. GAGA over the affinoid charts then checks pseudocoherence after a finite fpqc covering of schemes. Rational base change makes these constructions compatible over the analytic base.

We begin with the curve in Section 2, then construct the relative realizations and prove proper finiteness in Sections 3–4. The boundary calculation and GAGA are proved in Section 5. Section 6 gives the ordinary coherent comparisons.

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2. THE BASE CURVE AND ITS SOLID QUASICOHERENT CATEGORY

We first extend the comparison on the curve to quasicoherent complexes. The two-affinoid cover gives the compactness needed for full faithfulness and the cohomological bound needed for essential surjectivity on pseudocoherent objects.

2.1. A two-affinoid cover of the curve. We use the two-affinoid cover of [Ked17, Remark 3.1.9]. Its intersection has two components, which we describe explicitly.

Let $\mathcal{Y}$ be the adic period-ring space, let $\pi : \mathcal{Y} \rightarrow \mathcal{X} = \mathcal{Y}/\varphi^{\mathbf{Z}}$ be the quotient map, and let $\text{rad} : |\mathcal{Y}| \rightarrow (0, \infty)$ be the radius function. Choose compatible pseudo-uniformizers ϖ_E and ϖ_F . For a rational interval $I = [a, b] \subset (0, \infty)$, Fargues–Scholze [FS24, Definition II.1.15] define the rational affinoid

$$\mathcal{Y}_I = \{x \in \mathcal{Y} : |\varpi_E(x)|^b \leq |[\varpi_F](x)| \leq |\varpi_E(x)|^a\} \subset \text{rad}^{-1}(I),$$

and prove $\text{rad} \circ \varphi = q \text{ rad}$ [FS24, Definition II.1.15 and Proposition II.1.16]. Since $\varphi(\varpi_E) = \varpi_E$ and $\varphi([\varpi_F]) = [\varpi_F]^q$, the same defining inequalities give $\varphi(\mathcal{Y}_I) = \mathcal{Y}_{qI}$. We can now compute intersections directly from these inequalities.

Lemma 2.1. *Let $I = [a, b]$ and $J = [c, d]$ be rational intervals in $(0, \infty)$. Then*

$$\mathcal{Y}_I \cap \mathcal{Y}_J = \begin{cases} \mathcal{Y}_{[\max(a,c), \min(b,d)]}, & \max(a, c) \leq \min(b, d), \\ \emptyset, & \max(a, c) > \min(b, d). \end{cases}$$

Moreover, for every $n \in \mathbf{Z}$, $\varphi^n(\mathcal{Y}_I) = \mathcal{Y}_{q^n I}$. In particular, disjoint intervals define disjoint rational subspaces.

Proof. Intersecting the inequalities for I and J replaces their endpoints by $\max(a, c)$ and $\min(b, d)$. These inequalities have no solution when the first endpoint exceeds the second. Applying Frobenius multiplies both endpoints by q [FS24, Definition II.1.15 and Proposition II.1.16]; iteration gives the remaining assertion. $\square$

Lemma 2.2. *Let $U \subset \mathcal{Y}$ be an open analytic subspace satisfying $U \cap \varphi^n(U) = \emptyset$ for every $n \in \mathbf{Z} \setminus \{0\}$. Then the restriction $\pi|_U : U \rightarrow \mathcal{X}$ is an open immersion.*

Proof. The quotient by $\varphi^{\mathbf{Z}}$ is analytic because the action is free and totally discontinuous [FS24, Definition II.1.15 and Proposition II.1.16]. The saturation of U is

$$U^{\text{sat}} = \bigcup_{n \in \mathbf{Z}} \varphi^n(U) = \prod_{n \in \mathbf{Z}} \varphi^n(U).$$

It is open, and each of its orbits meets U once. Hence $\pi(U)$ is open and $|U| \rightarrow \pi(U)$ is a homeomorphism. Using U as the fundamental domain also identifies the restricted quotient groupoid with $U \rightrightarrows U$. Descent of the analytic structure gives $U \simeq \pi(U)$. $\square$

Proposition 2.3. *Choose rational numbers $1 < a' < a < b < b' < q$. Put $I_0 = [b/q, a]$ and $I_1 = [a', b']$, and let $\mathcal{U} = \pi(\mathcal{Y}_{I_0})$ and $\mathcal{V} = \pi(\mathcal{Y}_{I_1})$. Then π restricts to open immersions on $\mathcal{Y}_{I_0}$ and $\mathcal{Y}_{I_1}$, the two images cover $\mathcal{X}$, and*

$$\mathcal{U} \cap \mathcal{V} \simeq \mathcal{Y}_{[a', a]} \amalg \mathcal{Y}_{[b, b']}.$$

In the second component one uses φ to identify the representative in $\mathcal{Y}_{I_0}$ with the interval $[b, b'] \subset I_1$. Thus $\mathcal{X}$ has a cover by two affinoid opens whose intersection is a disjoint union of two affinoids. Each overlap component is a rational subdomain of both covering opens.

Proof. Both intervals have multiplicative width less than q :

$$\frac{a}{b/q} = q \frac{a}{b} < q,$$

$$\frac{b'}{a'} < q.$$

Their nontrivial Frobenius translates are disjoint. Lemmas 2.1 and 2.2 therefore give the two open immersions.

Every orbit meets $\mathcal{Y}_{[1,q]}$. We claim that

$$\mathcal{Y}_{[1,q]} = \mathcal{Y}_{[1,a]} \cup \mathcal{Y}_{[a',b']} \cup \mathcal{Y}_{[b,q]}.$$

Indeed, take $x \in \mathcal{Y}_{[1,q]}$ outside the first and third terms, and put $y = |[\varpi_F](x)|$. The defining inequalities give

$$|\varpi_E(x)|^{b'} < |\varpi_E(x)|^b < y < |\varpi_E(x)|^a < |\varpi_E(x)|^{a'}.$$

Thus $x \in \mathcal{Y}_{[a',b']}$. Since

$$\mathcal{Y}_{[1,a]} = \mathcal{Y}_{I_0} \cap \mathcal{Y}_{[1,q]},$$

$$\mathcal{Y}_{[b,q]} = \varphi(\mathcal{Y}_{I_0}) \cap \mathcal{Y}_{[1,q]},$$

and $\mathcal{Y}_{[a',b']} = \mathcal{Y}_{I_1}$, the images cover $\mathcal{X}$.

To compute the intersection, use representatives in $\mathcal{Y}_{I_1}$. Only $\mathcal{Y}_{I_0}$ and $\varphi(\mathcal{Y}_{I_0})$ meet this annulus: for $n \leq -1$, the upper endpoint of $q^n I_0$ is at most $a/q < a'$, and for $n \geq 2$, its lower endpoint is at least $qb > b'$. Lemma 2.1 now gives

$$\mathcal{Y}_{I_1} \cap \mathcal{Y}_{I_0} = \mathcal{Y}_{[a',a]},$$

$$\mathcal{Y}_{I_1} \cap \varphi(\mathcal{Y}_{I_0}) = \mathcal{Y}_{[b,b']}.$$

These are disjoint because $a < b$. Each inclusion is rational, being defined by the additional interval inequalities. On the second component, use φ^{-1} to obtain the inclusion into $\mathcal{Y}_{I_0}$. $\square$

2.2. A subordinate affinoid cover. Let $\text{RatAff}_X(\mathcal{X})$ be the basis of strongly noetherian rational affinoids $V \subset \lambda^{-1}(W)$, with W a standard affine open of X . This basis is stable under rational localization, and its intersections have finite covers by members of the basis.

Choose a finite standard affine cover (W_j) of X . Refine each of $\mathcal{U}$ and $\mathcal{V}$ by a finite rational-affinoid cover subordinate to the opens $\lambda^{-1}(W_j)$. This gives

$$\mathcal{X} = \bigcup_{i=1}^N V_i$$

with $V_i \subset \lambda^{-1}(W_{j(i)})$. We call this the subordinate cover. An intersection of its members lying in the same original annulus is rational affinoid. For a mixed intersection, restrict to the two components of $\mathcal{U} \cap \mathcal{V}$. On each component, all the restrictions are rational, so their intersection is rational affinoid. Thus every intersection is empty or a disjoint union of at most two strongly noetherian affinoids. Each lies over a standard affine of X .

Both this cover and the original two-open cover satisfy Proposition B.6.

2.3. Flatness and the comparison functor.

Lemma 2.4. *Let $W = \operatorname{Spec} A \subset X$ be a standard affine open, and let $V = \operatorname{Spa}(R, R^+) \subset \lambda^{-1}(W)$ be a strongly noetherian affinoid open. Then $A \rightarrow R$ is flat. Consequently, the morphism of locally ringed spaces $\lambda : \mathcal{X} \rightarrow X$ is flat on local rings.*

Proof. By the ideal criterion, it suffices to check that $I \otimes_A R \rightarrow R$ is injective for every finitely generated ideal $I \subset A$.

Extend $\tilde{I}$ to a coherent ideal $\mathcal{J} \subset \mathcal{O}_X$ [Stacks, Section 28.23, especially Lemma 28.23.2]. Coherent GAGA is exact [KL19, Theorem 4.7.4], so $\lambda^* \mathcal{J} \rightarrow \mathcal{O}_X$ is injective. Restrict to V and take sections. The finite-module theorem [Zav25, Theorem 4.2] identifies the resulting map with

$$I \otimes_A R \hookrightarrow R.$$

Thus $A \rightarrow R$ is flat.

For $x \in V$, the map $R \rightarrow \mathcal{O}_{V,x}$ is flat by [Zav25, Lemma 6.7], applied to the relative analytification of $\operatorname{Spec} R$. Composing with $A \rightarrow R$ and localizing at $\lambda(x)$ proves flatness on local rings. $\square$

We now define $\alpha_X : D_{\text{qc}}(X) \rightarrow \mathcal{C}_{\square}(\mathcal{X})$. For $M \in D_{\text{qc}}(X)$, $W = \operatorname{Spec} A \subset X$, and $V = \operatorname{Spa}(R, R^+) \subset \lambda^{-1}(W)$, put

$$\alpha_{W,V}(M) = \operatorname{Cond}_R(R \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(W, M|_W)).$$

This is independent of W . Indeed, if V lies over W_1 and W_2 , their intersection $W_{12} = \operatorname{Spec} A_{12}$ is affine, and quasicohherent restriction gives

$$A_{12} \otimes_{A_i}^{\mathbf{L}} \mathbf{R}\Gamma(W_i, M) \simeq \mathbf{R}\Gamma(W_{12}, M).$$

Tensoring with R identifies both definitions. For a rational inclusion $V' = \operatorname{Spa}(R', R'^+) \subset V$, scalar-extension compatibility of condensification gives

$$\alpha_{W,V}(M)|_{V'} \simeq \operatorname{Cond}_{R'}(R' \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(W, M)).$$

All comparison maps are tensor-associativity maps, so they satisfy the identity, composition, and higher coherences. Rational descent therefore gives $\alpha_X(M)$, functorially in M , and the local monoidal structures descend as well.

The local functors are exact and preserve filtered colimits [And21, Theorem 5.9 and Lemma 5.10]. They consequently preserve coproducts, by expressing a coproduct as the filtered colimit of its finite partial sums, and hence all small colimits. Colimits in the descent category are computed locally, so α_X has the same properties.

2.4. Enhanced full faithfulness.

Lemma 2.5. *For every bounded coherent complex K on X , the natural maps*

$$\mathbf{R}\Gamma_{\square}(\mathcal{X}, \alpha_X K) \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{adic}}(\mathcal{X}, \lambda^* K) \xleftarrow{\sim} \mathbf{R}\Gamma(X, K)$$

are equivalences.

Proof. For a coherent sheaf $\mathcal{F}$, the second comparison is [KL19, Theorem 4.7.5]. To pass to $K \in D_{\text{coh}}^b(X)$, choose an interval $[a, b]$ containing its cohomology and put $K_r = \tau_{\leq r} K$. The Postnikov triangles are

$$K_{r-1} \longrightarrow K_r \longrightarrow \mathcal{H}^r(K)[-r] \longrightarrow K_{r-1}[1].$$

Flatness of λ makes pullback exact and compatible with these truncations. Applying the algebraic and analytic section functors gives a map of exact triangles. The map on $\mathcal{H}^r(K)[-r]$ is an equivalence by coherent GAGA. Therefore the comparison for K_{r-1} implies the comparison for K_r . Starting at $K_{a-1} = 0$ and ending at $K_b = K$ proves the second map.

For the first map, use the subordinate cover from Section 2.2. On an affinoid component $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$ of an intersection, with $W = \mathrm{Spec} A$, set

$$K_R = R \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(W, K).$$

By Lemma 2.4, this has finite cohomology in $[a, b]$. Full faithfulness of condensification gives

$$\mathbf{R}\Gamma_{\square}(V, \mathrm{Cond}_R K_R) \simeq \mathrm{Map}_{\mathbf{D}(R)}(R, K_R) \simeq K_R$$

[And21, Theorem 5.9]. Ordinary affinoid acyclicity gives the same answer for a finite module, and the Postnikov induction above gives it for K_R .

Under a rational restriction $R \rightarrow R'$, both identifications use $K_R \rightarrow R' \otimes_R^{\mathbf{L}} K_R$. They therefore agree on overlaps. Apply finite Mayer–Vietoris to this cover, using Proposition B.6 on the solid side and [Stacks, Tag 08BX] on the ordinary side. The local equivalences give the first comparison on $\mathcal{X}$. $\square$

Theorem 2.6. *The functor $\alpha_X : \mathbf{D}_{\mathrm{qc}}(X) \rightarrow \mathcal{C}_{\square}(\mathcal{X})$ is fully faithful. It sends perfect complexes to compact objects, and its essential image is closed under arbitrary colimits.*

Proof. Let $P, Q \in \mathrm{Perf}(X)$. Duality and Lemma 2.5 give

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}_{\square}(\mathcal{X})}(\alpha_X P, \alpha_X Q) &\simeq \mathbf{R}\Gamma_{\square}(\mathcal{X}, \alpha_X(P^{\vee} \otimes Q)) \\ &\simeq \mathbf{R}\Gamma(X, P^{\vee} \otimes Q) \simeq \mathrm{Map}_{\mathbf{D}_{\mathrm{qc}}(X)}(P, Q). \end{aligned}$$

Thus the functor is fully faithful on perfect complexes. Moreover,

$$\mathrm{Map}(\alpha_X P, -) \simeq \mathbf{R}\Gamma_{\square}(\mathcal{X}, \alpha_X(P^{\vee}) \otimes -)$$

preserves filtered colimits by Proposition B.6. Hence $\alpha_X P$ is compact.

Now use $\mathbf{D}_{\mathrm{qc}}(X) \simeq \mathrm{Ind}(\mathrm{Perf}(X))$ [Stacks, Theorem 36.15.3, Tag 09IS, and Proposition 36.17.1, Tag 09M0]. Full faithfulness on perfect complexes and compactness of their images give full faithfulness of the colimit-preserving extension. This extension is α_X .

Finally, full faithfulness lifts any diagram in the image to a diagram (M_i) on X . Its colimit is again in the image, since

$$\mathrm{colim}_i \alpha_X(M_i) \simeq \alpha_X(\mathrm{colim}_i M_i).$$

$\square$

2.5. A finite fpqc affinoid family. The coordinate rings of a finite affinoid cover of $\mathcal{X}$ give the following fpqc morphism to X . We will use its schematic base changes to test pseudocoherence.

Lemma 2.7. *There are finitely many nonempty strongly noetherian rational affinoid opens $V_i = \mathrm{Spa}(R_i, R_i^+) \subset \mathcal{X}$ for $1 \leq i \leq m$, and standard affine opens $W_i = \mathrm{Spec} A_i \subset X$ with $V_i \subset \lambda^{-1}(W_i)$, such that the induced morphism*

$$s : \coprod_{i=1}^m \mathrm{Spec} R_i \longrightarrow X$$

is fpqc.

Proof. Choose a finite rational-affinoid cover (V_i) of $\mathcal{X}$ subordinate to a finite standard affine cover of X , and write $V_i \subset \lambda^{-1}(W_i)$ with $W_i = \operatorname{Spec} A_i$. This is possible by Proposition 2.3. Lemma 2.4 makes $A_i \rightarrow R_i$ flat. Since $W_i \hookrightarrow X$ is affine, each $\operatorname{Spec} R_i \rightarrow X$ is affine and flat.

We check surjectivity. Every nonempty V_i has $R_i \neq 0$. Flatness over the domain A_i makes R_i torsion-free, so

$$\operatorname{Frac}(A_i) \otimes_{A_i} R_i \neq 0.$$

A prime of this ring gives a point above the generic point of X .

For a closed point $x \in X$, coherent GAGA sends the nonzero skyscraper $\kappa(x)$ to a nonzero coherent sheaf [KL19, Theorem 4.7.4]. Its restriction to some V_i is nonzero. Thus $x \in W_i$ and

$$\kappa(x) \otimes_{A_i} R_i \neq 0.$$

A prime of this ring lies above x . Since X is an integral noetherian curve, these are all its points. The finite disjoint union is therefore surjective, flat, and quasi-compact, as required. $\square$

Put $\mathcal{B} = \operatorname{RatAff}_X(\mathcal{X})$ and $R_V = \Gamma(V, \mathcal{O}_V)$. Rational restriction gives the diagram

$$\begin{aligned} \mathcal{B}^{\text{op}} &\longrightarrow \operatorname{Cat}_{\infty}, \\ V &\longmapsto \operatorname{D}_{\text{pc}}(R_V), \end{aligned}$$

with transition functor $M \mapsto R_{V'} \otimes_{R_V}^{\mathbf{L}} M$ for $V' \subset V$. Write $\operatorname{Cart}_{V \in \mathcal{B}} \operatorname{D}_{\text{pc}}(R_V)$ for its limit.

Concretely, an object is a family $M_V \in \operatorname{D}_{\text{pc}}(R_V)$ with equivalences

$$\theta_{V',V} : R_{V'} \otimes_{R_V}^{\mathbf{L}} M_V \xrightarrow{\sim} M_{V'}.$$

For $V'' \subset V' \subset V$, these satisfy $\theta_{V'',V} \simeq \theta_{V'',V'} \circ (1 \otimes \theta_{V',V})$ under tensor associativity, with the unit and higher homotopies included. A morphism is a family of maps $M_V \rightarrow N_V$ commuting with these identifications and homotopies. Each M_V is an ordinary pseudocoherent complex.

Proposition 2.8. *For $V = \operatorname{Spa}(R, R^+) \in \operatorname{RatAff}_X(\mathcal{X})$, regard $\operatorname{D}_{\text{pc}}(R)$ as the full subcategory of nuclear and pseudocompact objects of $\operatorname{D}_{\square}(R, R^+)$ through condensification. Restriction induces an equivalence*

$$\operatorname{D}_{\text{pc}}^{\square}(\mathcal{X}) \xrightarrow{\sim} \operatorname{Cart}_{V \in \operatorname{RatAff}_X(\mathcal{X})} \operatorname{D}_{\text{pc}}(R).$$

The resulting full subcategory of $\mathcal{C}_{\square}(\mathcal{X})$ is independent of the rational basis and of the bounded-chart presentations.

Proof. On $V = \operatorname{Spa}(R, R^+)$, the affinoid comparison gives

$$\operatorname{Cond}_R : \operatorname{D}_{\text{pc}}(R) \xrightarrow{\sim} \operatorname{D}_{\text{pc}}^{\square}(V) \subset \operatorname{D}_{\square}(R, R^+)$$

[And21, Theorem 5.50] [LH26, A.15]. For a rational inclusion $u : V' \hookrightarrow V$, we have

$$u^* \operatorname{Cond}_R(M) \simeq \operatorname{Cond}_{R'}(R' \otimes_R^{\mathbf{L}} M)$$

[And21, Lemma 5.10]. Scalar extension preserves pseudocoherence, and these maps agree under composition. Thus condensification identifies the two diagrams of local categories.

Take their limits. Proposition B.1 identifies the ambient limit with $\mathcal{C}_{\square}(\mathcal{X})$. A cartesian section in the displayed full subcategories descends there, and its restrictions place it in

$D_{\text{pc}}^{\square}(\mathcal{X})$. Conversely, every object of $D_{\text{pc}}^{\square}(\mathcal{X})$ restricts to such a section. The same limit identifies maps, proving the equivalence.

Different rational bases admit common covering refinements. Pseudocoherent descent [And21, Theorem 5.3] compares the resulting diagrams, since the strongly noetherian pairs are sheafy. On every affinoid, the full subcategory is intrinsically $\text{Cond}_R(D_{\text{pc}}(R))$, so changing the bounded-chart presentation gives that same subcategory. This proves independence of the choices. $\square$

Lemma 2.9. *Let $Q \in D_{\text{pc}}^{\square}(\mathcal{X})$ and suppose that all its ordinary affinoid representatives have cohomology in degrees $\leq m$. For a vector bundle E on X , put $E^{\text{An}} = \alpha_X E$. Then, for every $i > m + 1$,*

$$H^i R\Gamma(\mathcal{X}, (E^{\text{An}})^{\vee} \otimes Q) = 0.$$

Proof. Use $\mathcal{X} = \mathcal{U} \cup \mathcal{V}$ from Proposition 2.3. On each of the two covering affinoids and each overlap component $W = \text{Spa}(B, B^+)$, let $Q_W \in D_{\text{pc}}(B)$ represent $Q|_W$. Such a representative exists by rational pseudocoherent descent: cover W by subordinate rational charts and apply [And21, Theorem 5.3]. Flat restriction gives $Q_W \in D^{\leq m}(B)$.

The bundle $(E^{\text{An}})^{\vee}|_W$ is represented by a finite projective module $E_W^{\vee}$. Hence

$$R\Gamma(W, (E^{\text{An}})^{\vee} \otimes Q) \simeq E_W^{\vee} \otimes_B^{\mathbf{L}} Q_W \in D^{\leq m}$$

[And21, Theorem 5.9 and Lemma 5.10]. By Lemma B.5, the global section complex is the fiber of the map from sections on $\mathcal{U}$ and $\mathcal{V}$ to sections on their intersection. Both terms are in $D^{\leq m}$, so the fiber is in $D^{\leq m+1}$. $\square$

We can now algebraize a pseudocoherent object. We will successively remove its highest cohomology sheaf using vector bundles, then take the colimit of the resulting perfect approximations.

Proposition 2.10. *The pseudocoherent subcategory $D_{\text{pc}}^{\square}(\mathcal{X})$ carries the t -structure obtained by rational descent from ordinary pseudocoherent complexes on the strongly noetherian affinoid charts. Analytification restricts to a t -exact equivalence*

$$\alpha_X : D_{\text{pc}}(X) \xrightarrow{\sim} D_{\text{pc}}^{\square}(\mathcal{X}).$$

Proof. For $C \in D_{\text{pc}}^{\square}(\mathcal{X})$, write $C_V \in D_{\text{pc}}(R)$ for its representative on $V = \text{Spa}(R, R^+)$. Define

$$\begin{aligned} D_{\text{pc}}^{\square}(\mathcal{X})^{\leq 0} &= \{C : C_V \in D_{\text{pc}}(R)^{\leq 0} \text{ for every } V\}, \\ D_{\text{pc}}^{\square}(\mathcal{X})^{\geq 0} &= \{C : C_V \in D_{\text{pc}}(R)^{\geq 0} \text{ for every } V\}. \end{aligned}$$

Rational localizations are flat. Thus ordinary truncations commute with restriction, and the triangles

$$\tau_{\leq 0} C_V \longrightarrow C_V \longrightarrow \tau_{\geq 1} C_V$$

form cartesian sections. Their compatibility under composition follows from the universal property of truncation. Proposition 2.8 descends these triangles. Orthogonality follows by taking the limit of the local mapping spaces, which are contractible in the required degrees. This gives the asserted t -structure [LH26, A.15] [And21, Theorems 5.44 and 5.50]. Its heart is $\text{Coh}(\mathcal{X})$, by finite-module descent.

On a subordinate chart, α_X is flat scalar extension followed by condensification. It therefore preserves pseudocoherence and is t -exact. Theorem 2.6 already gives full faithfulness.

Let $C \in \mathbf{D}_{\text{pc}}^\square(\mathcal{X})$. A finite affinoid cover gives a common upper bound b for its cohomology. We will approximate C by perfect complexes, removing one cohomology sheaf at a time.

First take $Q \in \mathbf{D}_{\text{pc}}^\square(\mathcal{X})^{\leq m}$. The curve X has the resolution property: it is quasi-compact and regular with affine diagonal [Stacks, Tag 0F8A]. Coherent GAGA therefore supplies a vector bundle E on X and a surjection

$$E^{\text{An}} \twoheadrightarrow \mathcal{H}^m(Q).$$

Put $F = \tau_{\leq m-1}Q$ and consider

$$F \longrightarrow Q \longrightarrow \mathcal{H}^m(Q)[-m] \longrightarrow F[1].$$

The obstruction to lifting the surjection to $E^{\text{An}}[-m] \rightarrow Q$ lies in

$$\text{Hom}(E^{\text{An}}[-m], F[1]) = H^{m+1} \mathbf{R}\Gamma(\mathcal{X}, (E^{\text{An}})^\vee \otimes F) = 0$$

by Lemma 2.9. A lift therefore exists, and its cofiber belongs to $\mathbf{D}_{\text{pc}}^\square(\mathcal{X})^{\leq m-1}$.

Start with $A_0 = 0 \rightarrow C$ and $Q_0 = C$. Suppose that $A_n \rightarrow C$ has perfect algebraic source and cofiber $Q_n \in \mathbf{D}_{\text{pc}}^\square(\mathcal{X})^{\leq b-n}$. The preceding construction gives

$$F_n = \alpha_X(E_n)[-(b-n)] \longrightarrow Q_n$$

surjective on $\mathcal{H}^{b-n}$. Define A_{n+1} by

$$\begin{array}{ccc} A_{n+1} & \longrightarrow & F_n \\ \downarrow & & \downarrow \\ C & \longrightarrow & Q_n. \end{array}$$

Then

$$\begin{array}{ccccc} A_n & \longrightarrow & A_{n+1} & \longrightarrow & F_n, \\ A_{n+1} & \longrightarrow & C & \longrightarrow & Q_{n+1} \end{array}$$

are cofiber sequences, with $Q_{n+1} \in \mathbf{D}_{\text{pc}}^\square(\mathcal{X})^{\leq b-n-1}$. The connecting map $F_n \rightarrow A_n[1]$ lifts by full faithfulness, so A_{n+1} again has a perfect algebraic preimage. Taking the pullbacks over C gives a sequential diagram. Lemma B.2 lifts it to $P_\bullet$ in $\text{Perf}(X)$.

Set $M = \text{colim}_n P_n$. Since α_X preserves colimits, the cofiber of $\alpha_X(M) \rightarrow C$ is $\text{colim}_n Q_n$. On any rational affinoid $V = \text{Spa}(R, R^+)$, lift the latter sequence to ordinary complexes $Q_{n,V} \in \mathbf{D}^{\leq b-n}(R)$. For each fixed j , we have

$$H^j(\text{colim}_n Q_{n,V}) = \text{colim}_n H^j(Q_{n,V}) = 0.$$

Indeed, the terms vanish for large n , and filtered colimits of modules are exact. Condensification preserves this colimit [And21, Theorem 5.9], so $(\text{colim}_n Q_n)|_V = 0$. Rational descent now gives $\alpha_X(M) \simeq C$.

It remains to show that M is pseudocoherent. Pull back along the fpqc morphism $s : \coprod_i \text{Spec } R_i \rightarrow X$ of Lemma 2.7. On V_i , the comparison reads

$$\text{Cond}_{R_i}(\mathbf{L}s_i^* M) \simeq C|_{V_i} \simeq \text{Cond}_{R_i}(P_i)$$

for some $P_i \in \mathbf{D}_{\text{pc}}(R_i)$. Full faithfulness gives $\mathbf{L}s_i^* M \simeq P_i$. Over an affine $W = \text{Spec } A \subset X$, the base change of s is a finite affine fpqc family and hence a single faithfully flat map $A \rightarrow A'$, using the product of its coordinate rings. The pullback of $M|_W$ to A' is pseudocoherent.

Faithfully flat descent [Stacks, Tag 068R] gives $M|_W \in D_{\text{pc}}(A)$. This proves essential surjectivity and the proposition. $\square$

3. RELATIVE REALIZATIONS AND RELATIVE FULL FAITHFULNESS

For a scheme over X , we construct the realizations on the affinoid charts of $\mathcal{X}$. On an affine morphism, the comparison is simply α_X applied to modules over the coordinate algebra. Zariski descent then gives the functor on the whole scheme.

3.1. Relative algebraic and analytic realizations. We use the basis $\mathcal{B} = \text{RatAff}_X(\mathcal{X})$ from Section 2.2.

Let T be a separated finitely presented X -scheme. Choose an affinoid $V = \text{Spa}(R, R^+) \subset \mathcal{X}$ whose image is contained in an affine $W = \text{Spec } A \subset X$, and set $T_R = T \times_X \text{Spec } R$. The construction of [LH26, Theorem A.13] gives the algebraic realization $(T_R)^{\text{Alg}}$ over $(\text{Spec } R)^{\text{Alg}}$ and its analytic locus $(T_R)^{\text{An}}$ over $V = (\text{Spec } R)^{\text{An}}$. Define locally

$$\begin{aligned} (T_{\mathcal{X}}^{\text{Alg}})|_V &= (T_R)^{\text{Alg}} \times_{(\text{Spec } R)^{\text{Alg}}} V, \\ (T_{\mathcal{X}}^{\text{An}})|_V &= (T_R)^{\text{An}}. \end{aligned}$$

Lemma 3.1. *Let $u : V' = \text{Spa}(R', R'^+) \rightarrow V = \text{Spa}(R, R^+)$ be a rational localization between members of $\text{RatAff}_X(\mathcal{X})$. For every separated finitely presented X -scheme T , there are natural equivalences*

$$\begin{aligned} u^*((T_R)^{\text{Alg}} \times_{(\text{Spec } R)^{\text{Alg}}} V) &\simeq (T_{R'})^{\text{Alg}} \times_{(\text{Spec } R')^{\text{Alg}}} V', \\ u^*(T_R)^{\text{An}} &\simeq (T_{R'})^{\text{An}}. \end{aligned}$$

They are functorial in T , compatible with Zariski restriction, and satisfy the identity and composition cocycles for chains of rational localizations.

Proof. The scheme $T_R = T \times_X \text{Spec } R$ is intrinsic to the map $\text{Spec } R \rightarrow X$. Two affine neighborhoods used to describe that map have an affine intersection, on which the descriptions agree.

If $T_R = \text{Spec } B$, the algebraic comparison is Lemma B.9(i) after applying AnSpec , and the analytic comparison is part (iii). For general T_R , choose a finite affine cover (T_a) . Its intersections are affine since T_R is separated. By [LH26, Theorem A.13],

$$\begin{aligned} (T_R)^{\text{Alg}} &\simeq \text{colim}_{[n] \in \Delta^{\text{op}}} \coprod_{(a_0, \dots, a_n)} (T_{a_0 \dots a_n})^{\text{Alg}}, \\ (T_R)^{\text{An}} &\simeq \text{colim}_{[n] \in \Delta^{\text{op}}} \coprod_{(a_0, \dots, a_n)} (T_{a_0 \dots a_n})^{\text{An}}. \end{aligned}$$

Pullback in the slice topos preserves colimits. We can therefore apply the affine comparisons to this diagram and take its realization. The maps commute with the face and degeneracy maps, so their functoriality and composition compatibilities pass to the realization. A common affine refinement compares different covers. $\square$

Proposition 3.2. *Let T be a separated finitely presented X -scheme. The local algebraic and analytic realizations just defined glue to analytic stacks over $\mathcal{X}$, denoted $T_{\mathcal{X}}^{\text{Alg}}$ and $T_{\mathcal{X}}^{\text{An}}$. The construction is functorial in T , commutes with rational open base change on $\mathcal{X}$, and is compatible with Zariski descent in T . If $T \hookrightarrow X$ is an open immersion, $T_{\mathcal{X}}^{\text{An}}$ identifies with*

the analytic open $\lambda^{-1}(T) \subset \mathcal{X}$. The local analytic-locus maps glue to a natural morphism $T_{\mathcal{X}}^{\text{An}} \rightarrow T_{\mathcal{X}}^{\text{Alg}}$, which is an isomorphism when $T \rightarrow X$ is proper.

Proof. Lemma 3.1 gives compatible pullback identifications for

$$\begin{aligned} V &\mapsto (T_R)^{\text{Alg}} \times_{(\text{Spec } R)^{\text{Alg}}} V, \\ V &\mapsto (T_R)^{\text{An}}. \end{aligned}$$

Descent for analytic stacks glues these to the required realizations. Applying descent to their maps gives functoriality in T , and the Zariski gluing in [LH26, Theorem A.13] gives compatibility with a Zariski cover of T .

For an open $T \subset X$, each analytic realization is the inverse-image open, so the global realization is $\lambda^{-1}(T)$. The map $T_{\mathcal{X}}^{\text{An}} \rightarrow T_{\mathcal{X}}^{\text{Alg}}$ is obtained in the same way: on affines use the bounded-locus map of [LH26, A.11–A.13], then glue over a Zariski cover and over the rational base charts. Lemma B.9 gives the required compatibility.

If $T \rightarrow X$ is proper, every $T_R \rightarrow \text{Spec } R$ is proper. The local analytic-locus map is then an equivalence by [LH26, Theorem A.13]. It follows by descent that the global map is an equivalence. $\square$

For the relative space, set $T_R = T \times_X \text{Spec } R$ and $\mathcal{P}_T(V) = \mathbf{D}_{\text{pc}}^{\square}((T_R)^{\text{An}})$. Each value is a category on the whole space above V , with nuclearity and pseudocompactness tested on the bounded charts of [LH26, A.15]. For a rational inclusion $u : V' \hookrightarrow V$, Lemma 3.1 gives the pullback $u_T^* : \mathcal{P}_T(V) \rightarrow \mathcal{P}_T(V')$; on analytic affines this is completed derived scalar extension.

We write $\text{Cart}_{V \in \mathcal{B}} \mathcal{P}_T(V)$ for the limit. Concretely, its objects are families $C_V \in \mathcal{P}_T(V)$ with pullback equivalences $u_T^* C_V \simeq C_{V'}$, unit and composition homotopies, and their higher compatibilities. Its maps are families of maps commuting with this data. Each C_V is an object on the entire analytic space $(T_R)^{\text{An}}$.

Proposition 3.3. *Let T be a separated finitely presented X -scheme. Restriction induces an equivalence*

$$\mathbf{D}_{\text{pc}}^{\square}(T_{\mathcal{X}}^{\text{An}}) \xrightarrow{\sim} \text{Cart}_{V \in \text{RatAff}_X(\mathcal{X})} \mathbf{D}_{\text{pc}}^{\square}((T_R)^{\text{An}}).$$

The transition functors are the rational pullbacks of Lemma 3.1. This description is independent of the rational basis and of all bounded-chart presentations. If T is proper, the same description is transported to $T_{\mathcal{X}}^{\text{Alg}}$ through the analytic-locus equivalence.

Proof. We first check the transition functors. For scalar extension $F : \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A}')$, let G be restriction of scalars. It preserves direct sums and is left t -exact in our cohomological convention [Sch26a, Proposition 12.4, Remark 12.5, and Proposition 12.6]. Thus, for M pseudocompact and (N_a) uniformly bounded below,

$$\begin{aligned} \text{Map}(FM, \bigoplus_a N_a) &\simeq \text{Map}(M, \bigoplus_a GN_a) \\ &\simeq \bigoplus_a \text{Map}(M, GN_a) \simeq \bigoplus_a \text{Map}(FM, N_a). \end{aligned}$$

This proves pseudocompactness of FM by [LH26, A.5]. Scalar extension also preserves nuclear objects: it is strong symmetric monoidal, preserves colimits, and takes trace-class presentations to trace-class presentations [LH26, A.4]. Its compact-object preservation follows because G preserves filtered colimits.

Fix $V = \mathrm{Spa}(R, R^+)$ and an affine intersection $\mathrm{Spec} B$ in a finite Zariski cover of T_R . Choose generators of B . The universal bounded charts used for pseudocompactness commute with rational base change by Lemma B.9, so the preceding argument applies to them.

For nuclearity, the second chart system in [LH26, A.15] uses the analytic ring $\mathcal{Q}_R$ induced along

$$R \longrightarrow R\langle T_1, \dots, T_m \rangle$$

[LH26, A.10]. The continuous map of Tate algebras for $R \rightarrow R'$ induces $\mathcal{Q}_R \rightarrow \mathcal{Q}_{R'}$. Indeed, a complete $\mathcal{Q}_{R'}$ -module is complete over $(R', R'^+)^{\square}$, hence over $(R, R^+)^{\square}$; the induced-structure criterion then makes its restriction $\mathcal{Q}_R$ -complete [Sch26a, Proposition 12.8] [LH26, A.3]. Combining this map with algebraic base change gives a map of the pushouts defining the nuclearity charts. The construction also applies to the scaled generators $\varpi^n b_i$. Pullback on these charts preserves nuclearity by the affine argument above.

The two finiteness conditions descend over these charts [LH26, A.15]. They also descend over the Zariski cover, whose realization is a cover by cohomologically étale steady localizations [LH26, Theorem A.13]. Proposition B.1 therefore gives pullback functors on $\mathcal{P}_T(V)$. The affine base-change maps commute with restriction, and associativity of the pushouts gives their coherent composition. We obtain the diagram $\mathcal{P}_T : \mathcal{B}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$.

Now take the limit. Rational descent gives

$$\mathcal{C}_{\square}(T_{\mathcal{X}}^{\mathrm{An}}) \simeq \lim_{V \in \mathcal{B}^{\mathrm{op}}} \mathcal{C}_{\square}((T_R)^{\mathrm{An}}).$$

A cartesian section in the pseudocoherent subcategories descends in these ambient categories. Its local restrictions are nuclear and pseudocompact, so the descended object belongs to $\mathrm{D}_{\mathrm{pc}}^{\square}(T_{\mathcal{X}}^{\mathrm{An}})$. Conversely, any object of this category gives such a section. The same descent identifies the mapping spaces.

Common rational and Zariski refinements compare choices of covers. Independence of the bounded presentations follows from descent for the two finiteness conditions, with the pullback compatibility just proved. For proper T , Proposition 3.2 identifies the analytic and algebraic realizations over every chart, giving the last assertion. $\square$

Lemma 3.4. *Let T be a separated finitely presented X -scheme and let $(U_i)_{i \in I}$ be a finite Zariski open cover. Put $U_{i_0 \dots i_n} = U_{i_0} \cap \dots \cap U_{i_n}$. Restriction induces canonical equivalences*

$$\begin{aligned} \mathrm{D}_{\mathrm{qc}}(T) &\simeq \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n) \in I^{n+1}} \mathrm{D}_{\mathrm{qc}}(U_{i_0 \dots i_n}), \\ \mathcal{C}_{\square}(T_{\mathcal{X}}^{\mathrm{Alg}}) &\simeq \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n) \in I^{n+1}} \mathcal{C}_{\square}(U_{i_0 \dots i_n}^{\mathrm{Alg}}). \end{aligned}$$

The relative comparison functors form a natural transformation between these cosimplicial diagrams. Degree-wise full faithfulness, equivalence, or colimit preservation therefore descends to the two limits. Symmetric monoidal structures, module objects, and pullback transformations descend through the same diagrams.

Proof. The first formula is Zariski descent for quasicoherent complexes. By [LH26, Theorem A.13], the algebraic realization is the geometric realization of the corresponding simplicial diagram. Lemma 3.1 preserves that diagram under rational base change. Proposition B.1 now turns its realization into the second displayed limit.

The comparison functors commute with the restriction maps. Apply Lemma B.2 to obtain full faithfulness, equivalence, or colimit preservation as appropriate. The tensor structures and module objects are computed in the same limits by Proposition B.1. $\square$

3.2. Ordinary relative Huber analytification. Huber's mapping property gives the ordinary relative analytification and its compatibility with the base charts.

Proposition 3.5. *Let T be a finitely presented X -scheme. For $V = \mathrm{Spa}(R, R^+)$, put $T_R = T \times_X \mathrm{Spec} R$. The local relative Huber analytifications*

$$(T_R)^{\mathrm{ad}/V} \longrightarrow V$$

glue canonically to a locally strongly noetherian adic space $T_{\mathcal{X}}^{\mathrm{ad}}$ over $\mathcal{X}$, equipped with a morphism of locally ringed spaces $c_T : T_{\mathcal{X}}^{\mathrm{ad}} \longrightarrow T$. The construction commutes with rational base change on $\mathcal{X}$ and Zariski gluing in T . If $T \rightarrow X$ is proper, then $T_{\mathcal{X}}^{\mathrm{ad}} \rightarrow \mathcal{X}$ is proper.

Proof. For $V = \mathrm{Spa}(R, R^+) \in \mathrm{RatAff}_X(\mathcal{X})$, use Huber's mapping property to form $(T_R)^{\mathrm{ad}/V}$ [Hub94, Proposition 3.8]. For a rational $V' = \mathrm{Spa}(R', R'^+) \subset V$, that property gives

$$(T_R)^{\mathrm{ad}/V} \times_V V' \xrightarrow{\sim} (T_{R'})^{\mathrm{ad}/V'}.$$

For successive restrictions, the two composites represent the same identity transformation and therefore agree.

Choose a rational-affinoid cover (V_i) of $\mathcal{X}$. On an overlap, use common rational refinements to identify the two analytifications. Uniqueness makes these identifications agree on further refinements and triple overlaps. They glue the adic spaces and their maps to T , giving $T_{\mathcal{X}}^{\mathrm{ad}}$ and c_T . The same mapping property gives independence of the cover and compatibility with Zariski gluing in T .

If $T \rightarrow X$ is proper, each $(T_R)^{\mathrm{ad}/V} \rightarrow V$ is proper [Zav25, Remark 6.3]. Since properness is local on the target for open covers, so is the global map. $\square$

3.3. Module categories and relative full faithfulness. For an affine morphism, quasi-coherent complexes are modules over its coordinate algebra. We use this description to extend full faithfulness from X to the total space.

Let $q : U \rightarrow X$ be affine and of finite presentation, and put $\mathcal{A} = q_* \mathcal{O}_U \in \mathrm{CAlg}(\mathrm{D}_{\mathrm{qc}}(X))$. Then $\mathrm{D}_{\mathrm{qc}}(U) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{D}_{\mathrm{qc}}(X))$.

Lemma 3.6. *There is a canonical equivalence*

$$\mathcal{C}_{\square}(U_{\mathcal{X}}^{\mathrm{Alg}}) \simeq \mathrm{Mod}_{\alpha_X \mathcal{A}}(\mathcal{C}_{\square}(\mathcal{X})),$$

compatible with rational restriction on $\mathcal{X}$. Under this equivalence, $\mathbf{R}q_*^{\mathrm{Alg}}$ is the forgetful functor to $\mathcal{C}_{\square}(\mathcal{X})$, and the comparison functor on an object of $\mathrm{D}_{\mathrm{qc}}(U)$ is the module functor induced by α_X .

Proof. Work over $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$, and write $U_R = \mathrm{Spec} B$. Put $\mathcal{R} = (R, R^+)^{\square}$ and

$$\begin{aligned} \mathcal{R}_0 &= (R_{\mathrm{disc}}, R_{\mathrm{disc}}^+)^{\square}, \\ \mathcal{B}_0 &= (B_{\mathrm{disc}}, B_{\mathrm{disc}}^+)^{\square}, \end{aligned}$$

where B^+ is the integral closure of the image of R^+ in B . The algebraic realization is defined by $B_R^{\text{alg}} = \mathcal{R} \otimes_{\mathcal{R}_0}^{\text{AnRing}} \mathcal{B}_0$ [LH26, A.9]. Using Lemma B.7 and the module-category base-change formula, we compute

$$\begin{aligned} \mathrm{D}(B_R^{\text{alg}}) &\simeq \mathrm{D}(\mathcal{R}) \otimes_{\mathrm{D}(\mathcal{R}_0)} \mathrm{D}(\mathcal{B}_0) \\ &\simeq \mathrm{D}(\mathcal{R}) \otimes_{\mathrm{D}(\mathcal{R}_0)} \mathrm{Mod}_B(\mathrm{D}(\mathcal{R}_0)) \\ &\simeq \mathrm{Mod}_{\mathrm{Cond}_R(B)}(\mathrm{D}_{\square}(R, R^+)). \end{aligned}$$

The first and last steps use [LH26, Lemma A.1] and [Lur17, Section 4.8.5]. By the definition of α_X , the algebra $\mathrm{Cond}_R(B)$ is $(\alpha_X \mathcal{A})|_V$.

For a rational restriction $R \rightarrow R'$, Lemma B.8 identifies this calculation with the one for $B \otimes_R R'$. The identifications are associative, so Proposition B.1 glues the module categories. On each chart the structural pullback is the free-module functor and its right adjoint forgets the action. These descriptions glue too, proving both remaining assertions. $\square$

Proposition 3.7. *For every separated finitely presented X -scheme T , scalar extension induces a fully faithful colimit-preserving functor $\alpha_T : \mathrm{D}_{\mathrm{qc}}(T) \rightarrow \mathcal{C}_{\square}(T_{\mathcal{X}}^{\text{Alg}})$. It is strong symmetric monoidal and compatible with pullback. If $T \rightarrow X$ is proper, we use Proposition 3.2 to regard its target as $\mathcal{C}_{\square}(T_{\mathcal{X}}^{\text{An}})$.*

Proof. For an affine $T \rightarrow X$, Lemma 3.6 identifies the comparison with the functor on modules induced by α_X . Theorem 2.6 and Lemma B.3 give full faithfulness.

For general T , choose a finite affine cover (U_i) subordinate to a finite affine cover (W_a) of X . Each $U_i \rightarrow X$ is affine: $U_i \rightarrow W_a$ is a map of affine schemes, and $W_a \hookrightarrow X$ is affine because X is separated. Every multiple intersection U_I is also affine over X , by separatedness of T and the same argument over intersections of the W_a .

Apply the affine comparison to the full Čech nerve. The resulting functors are fully faithful, strong symmetric monoidal, and colimit-preserving, and they commute with restriction by tensor associativity. Taking the limits of Lemma 3.4 proves the result. Monoidal structures and pullback maps descend through this same diagram. $\square$

Proposition 3.8. *Let T be a separated finitely presented X -scheme, and let $j_T : T_{\mathcal{X}}^{\text{An}} \rightarrow T_{\mathcal{X}}^{\text{Alg}}$ be the analytic-locus morphism. The composite $\alpha_T^{\text{An}} := j_T^* \alpha_T : \mathrm{D}_{\mathrm{pc}}(T) \rightarrow \mathcal{C}_{\square}(T_{\mathcal{X}}^{\text{An}})$ has image in $\mathrm{D}_{\mathrm{pc}}^{\square}(T_{\mathcal{X}}^{\text{An}})$. If T is proper, j_T is an isomorphism, so α_T^{An} is the restriction of the fully faithful functor of Proposition 3.7; in that case we again write it simply as α_T .*

Proof. Take $M \in \mathrm{D}_{\mathrm{pc}}(T)$. Over a rational-affinoid base chart, choose an affine Zariski cover and the bounded analytic charts of [LH26, A.15]. Locally, M has a bounded-above resolution by finite free modules over a finite type R -algebra B . Derived scalar extension carries it to a bounded-above finite-free complex on each bounded chart.

Its condensification is nuclear and pseudocompact by [LH26, A.15 and the paragraph preceding Theorem A.18], using [And21, Theorem 5.50 and Proposition 5.33]. These local complexes are the restrictions of $j_T^* \alpha_T(M)$, by the scalar-extension construction. Descent for nuclearity and pseudocompactness therefore puts this object in $\mathrm{D}_{\mathrm{pc}}^{\square}(T_{\mathcal{X}}^{\text{An}})$. $\square$

3.4. Flatness.

Proposition 3.9. *For every finitely presented X -scheme T , the morphism $c_T : T_{\mathcal{X}}^{\text{ad}} \rightarrow T$ is flat.*

Proof. Take $W = \operatorname{Spec} A \subset X$, $V = \operatorname{Spa}(R, R^+) \subset \lambda^{-1}(W)$, and an affine $U \subset T \times_X W$. Over V , put $U_R = U \times_A \operatorname{Spec} R$. The map factors as

$$U_R^{\operatorname{ad}/V} \xrightarrow{c_{U_R/V}} U_R \longrightarrow U.$$

The second map is flat by Lemma 2.4; the first is flat on local rings by [Zav25, Lemma 6.7]. These charts cover the source and target, so c_T is flat. $\square$

4. PROPER PUSHFORWARD AND LOCALIZATION AT THE BOUNDARY

The next step is to write an affine restriction in terms of proper direct images. We first establish their finiteness, then localize along a boundary divisor.

4.1. Proper direct images with solid coefficients.

4.1.1. *The affinoid case.* We use the proper-direct-image argument of [LH26, A.16–A.17]. Over our strongly noetherian bases, the polynomial-module finiteness required in [LH26, A.14] follows from a finite-syzygy calculation.

Let (R, R^+) be a strongly noetherian complete Tate Huber pair. For a scheme S locally of finite type over R , write S^{An} for its analytification as an analytic stack [LH26, Theorem A.13] and $\pi : S^{\operatorname{An}} \rightarrow \operatorname{AnSpec}(R, R^+)^{\square}$ for its structural morphism. We use $\mathbf{D}_{\operatorname{pc}}^{\square}(S^{\operatorname{An}})$ for the category of nuclear and pseudocompact objects defined in [LH26, A.15], and the fully faithful condensification functor $\operatorname{Cond}_R : \mathbf{D}_{\operatorname{pc}}(R) \hookrightarrow \mathbf{D}_{\square}(R, R^+)$ of [And21, Theorem 5.9].

Theorem 4.1. *Let (R, R^+) be a strongly noetherian complete Tate Huber pair, and let S be a proper scheme of finite presentation over R . Then the solid direct-image functor $\mathbf{R}\pi_* : \mathcal{C}_{\square}(S^{\operatorname{An}}) \rightarrow \mathbf{D}_{\square}(R, R^+)$ sends $\mathbf{D}_{\operatorname{pc}}^{\square}(S^{\operatorname{An}})$ into the essential image of $\operatorname{Cond}_R : \mathbf{D}_{\operatorname{pc}}(R) \hookrightarrow \mathbf{D}_{\square}(R, R^+)$. Equivalently, for every $K \in \mathbf{D}_{\operatorname{pc}}^{\square}(S^{\operatorname{An}})$ there exists an object $M \in \mathbf{D}_{\operatorname{pc}}(R)$ and an equivalence $\operatorname{Cond}_R(M) \simeq \mathbf{R}\pi_* K$ in $\mathbf{D}_{\square}(R, R^+)$. The object M is unique up to a contractible space of choices.*

Proof. We prove the finiteness needed for the exceptional-pullback argument of [LH26, Proposition A.16 and Corollary A.17].

Let $P = R[T_1, \dots, T_m]$ and $B = P/I$. The ring P is noetherian, so successive finite-free presentations of B and its kernels give a bounded-above finite-free resolution $F^{\bullet} \rightarrow B$. Thus $B \in \mathbf{D}_{\operatorname{pc}}(P)$, as required in [LH26, Lemma A.14].

Put $D = R\langle T_1, \dots, T_m \rangle$. For a finitely generated ideal $J \subset P$, flatness of relative analytification on local rings [Zav25, Lemma 6.7] preserves the injection $\tilde{J} \rightarrow \mathcal{O}_{\mathbf{A}_R^m}$. Restricting to the closed unit polydisc and taking sections gives $J \otimes_P D \hookrightarrow D$, by [Zav25, Theorem 4.2]. The ideal criterion therefore makes $P \rightarrow D$ flat [Stacks, Tag 00HD], and

$$D \otimes_P^{\mathbf{L}} B \simeq (D/ID)[0] \in \mathbf{D}_{\operatorname{pc}}(D).$$

The tensor product of $F^{\bullet}$ with D gives its finite-free resolution. With the finite-module topology, this quotient condensifies to a nuclear and pseudocompact object [And21, Theorem 5.9, Lemma 5.23, and Theorem 5.50]. The same calculation applies to $R\langle \varpi^n \mathbf{T} \rangle$ and after rational localization of R .

We can now use [LH26, Proposition A.16]. On a quotient chart, the right adjoint is internal derived Hom from $(D/ID)[0]$. Its pseudocompactness gives direct-sum compatibility on uniformly bounded-below targets, and its concentration in degree zero gives the lower

bound. On a polynomial chart, exceptional pullback is ordinary pullback shifted by $[m]$. These calculations glue over the bounded charts and a finite affine cover of S . Hence, for $(M_a) \subset \mathbf{D}_\square(R, R^+)^{\geq d}$, the objects $\pi^! M_a$ have a common lower bound on the defining charts, and

$$\bigoplus_a \pi^! M_a \xrightarrow{\sim} \pi^! \left(\bigoplus_a M_a \right).$$

Properness gives $\mathbf{R}\pi_* = \mathbf{R}\pi_!$. The finite bounded-chart argument of [LH26, A.3, A.10, A.13, A.15, and the proof of Corollary A.17] shows that it preserves nuclearity. For pseudocompactness, take $K \in \mathbf{D}_{\text{pc}}^\square(S^{\text{An}})$ and a uniformly bounded-below family (M_a) . We have

$$\begin{aligned} \text{Map}(\mathbf{R}\pi_* K, \bigoplus_a M_a) &\simeq \text{Map}(K, \pi^! \bigoplus_a M_a) \\ &\simeq \text{Map}(K, \bigoplus_a \pi^! M_a) \\ &\simeq \bigoplus_a \text{Map}(K, \pi^! M_a) \\ &\simeq \bigoplus_a \text{Map}(\mathbf{R}\pi_* K, M_a). \end{aligned}$$

The third step uses the common lower bound and the test in [CS26, Definition 9.9]. Thus $\mathbf{R}\pi_* K$ is nuclear and pseudocompact. The affinoid comparison [LH26, A.15] [And21, Theorem 5.50] gives $\mathbf{R}\pi_* K \simeq \text{Cond}_R(M)$ for $M \in \mathbf{D}_{\text{pc}}(R)$. Full faithfulness gives uniqueness of M equipped with this equivalence. $\square$

Corollary 4.2. *In the setting of Theorem 4.1, the functor $\mathbf{R}\pi_* : \mathcal{C}_\square(S^{\text{An}}) \longrightarrow \mathbf{D}_\square(R, R^+)$ commutes with arbitrary colimits.*

Proof. For proper π , the proof above identifies $\mathbf{R}\pi_*$ with $\mathbf{R}\pi_!$ [LH26, proof of Corollary A.17]. The latter is left adjoint to $\pi^!$ and therefore preserves colimits. $\square$

4.1.2. Globalization. Let $g : Z \rightarrow X$ be proper and of finite presentation, with relative analytification $g^{\text{An}} : Z_{\mathcal{X}}^{\text{An}} \longrightarrow \mathcal{X}$.

We first check open base change, so that the affinoid direct images compute the restrictions of the global one.

Lemma 4.3. *Let $i : V \hookrightarrow \mathcal{X}$ be a quasi-compact analytic open embedding in the class I of [LH26, A.2]. This includes rational-affinoid open immersions. Form the cartesian square*

$$\begin{array}{ccc} Z_V^{\text{An}} & \xrightarrow{i'} & Z_{\mathcal{X}}^{\text{An}} \\ g_V^{\text{An}} \downarrow & & \downarrow g^{\text{An}} \\ V & \xrightarrow{i} & \mathcal{X}. \end{array}$$

Then the canonical transformation

$$i^* \mathbf{R}g_*^{\text{An}} \longrightarrow \mathbf{R}g_{V,*}^{\text{An}} i'^*$$

is an equivalence on $\mathcal{C}_\square(Z_{\mathcal{X}}^{\text{An}})$.

Proof. The class I is stable under base change, and its pullbacks admit exact left adjoints given by extension by zero [LH26, A.2]. Rational-affinoid open immersions belong to I by [RC24, Proposition 2.3.2]. The stack-level formalism [Sch26b, Proposition 5.16 and Theorem 5.19] gives the adjunctions $i_! \dashv i^*$ and $i'_! \dashv i'^*$ and the equivalence

$$(g^{\text{An}})^* i_! \simeq i'_! (g_V^{\text{An}})^*$$

with the base-change maps of [Sch26b, Definition 2.5].

For $A \in \mathcal{C}_\square(Z_\mathcal{X}^{\text{An}})$ and $B \in \mathcal{C}_\square(V)$, compute

$$\begin{aligned} \text{Map}_V(B, i^* \mathbf{R}g_*^{\text{An}} A) &\simeq \text{Map}_\mathcal{X}(i_! B, \mathbf{R}g_*^{\text{An}} A) \\ &\simeq \text{Map}_{Z_\mathcal{X}^{\text{An}}}((g^{\text{An}})^* i_! B, A) \\ &\simeq \text{Map}_{Z_\mathcal{X}^{\text{An}}}(i'_! (g_V^{\text{An}})^* B, A) \\ &\simeq \text{Map}_{Z_V^{\text{An}}}((g_V^{\text{An}})^* B, i'^* A) \\ &\simeq \text{Map}_V(B, \mathbf{R}g_{V,*}^{\text{An}} i'^* A). \end{aligned}$$

The adjunctions identify this equivalence with the mate of the given square. Yoneda proves the assertion for the canonical map. $\square$

Proposition 4.4. *Let $g : Z \rightarrow X$ be proper and of finite presentation. Then $\mathbf{R}g_*^{\text{An}} : \mathcal{D}_{\text{pc}}^\square(Z_\mathcal{X}^{\text{An}}) \rightarrow \mathcal{D}_{\text{pc}}^\square(\mathcal{X})$ preserves chartwise pseudocoherent objects. Moreover, the functor $\mathbf{R}g_*^{\text{An}} : \mathcal{C}_\square(Z_\mathcal{X}^{\text{An}}) \rightarrow \mathcal{C}_\square(\mathcal{X})$ commutes with arbitrary colimits.*

Proof. Take $V = \text{Spa}(R, R^+) \in \text{RatAff}_X(\mathcal{X})$ and put $Z_R = Z \times_X \text{Spec } R$. Proposition 3.2 and Lemma 4.3 give

$$(\mathbf{R}g_*^{\text{An}} K)|_V \simeq \mathbf{R}(g_R)_*^{\text{An}}(K|_{Z_R^{\text{An}}}).$$

For $K \in \mathcal{D}_{\text{pc}}^\square(Z_\mathcal{X}^{\text{An}})$, the local Grauert theorem puts the right side in $\text{Cond}_R(\mathcal{D}_{\text{pc}}(R))$. These are the restrictions of $\mathbf{R}g_*^{\text{An}} K$, so this object belongs to $\mathcal{D}_{\text{pc}}^\square(\mathcal{X})$.

For a diagram (K_a) , restrict the map

$$\text{colim}_a \mathbf{R}g_*^{\text{An}} K_a \rightarrow \mathbf{R}g_*^{\text{An}} \left(\text{colim}_a K_a \right)$$

to V . Open base change identifies it with the corresponding map for $(g_R)^{\text{An}}$, which is an equivalence by Corollary 4.2. Restriction to an affinoid cover is conservative, so the map is an equivalence globally. $\square$

4.2. Algebraic Cartier localization.

Lemma 4.5. *Let (R, R^+) be a complete Tate Huber pair, let S be a separated finitely presented R -scheme, let $\mathcal{L}$ be a line bundle on S , and let $s : \mathcal{O}_S \rightarrow \mathcal{L}$ be a section. Write $U = S_s \subset S$ for its nonvanishing locus and $j : U \hookrightarrow S$ for the open immersion. Use the same symbols $\mathcal{L}$ and s for their images under the comparison functor. Then $j^{\text{Alg},*} : \mathcal{C}_\square(S^{\text{Alg}}) \rightarrow \mathcal{C}_\square(U^{\text{Alg}})$ admits a right adjoint $\mathbf{R}j_*^{\text{Alg}}$, and for every $K \in \mathcal{C}_\square(S^{\text{Alg}})$ there is a natural equivalence*

$$\mathbf{R}j_*^{\text{Alg}} j^{\text{Alg},*} K \simeq \text{colim}_{n \geq 0} (K \otimes \mathcal{L}^{\otimes n}),$$

whose transition maps are induced by s . No regularity assumption on the section is required.

Proof. Trivialize $\mathcal{L}$ on a finite affine Zariski cover. On an intersection $\mathrm{Spec} A$, write $s = t$. The nonvanishing locus is $\mathrm{Spec} A[t^{-1}]$, and Lemma B.10 identifies pullback with $-\otimes_A^{\mathbf{L}} A[t^{-1}]$. Its right adjoint is restriction of scalars.

For a further restriction $A \rightarrow A'$, the identity

$$A'[t^{-1}] \simeq A' \otimes_A^{\mathbf{L}} A[t^{-1}]$$

gives the right-adjoint base-change equivalence

$$A' \otimes_A^{\mathbf{L}} \mathrm{Res}_{A[t^{-1}]/A} M \simeq \mathrm{Res}_{A'[t^{-1}]/A'} (A'[t^{-1}] \otimes_{A[t^{-1}]}^{\mathbf{L}} M).$$

Associativity makes these maps compatible on longer intersections. By Lemma B.4, the right adjoints glue through the Zariski presentation of S^{Alg} [LH26, Theorem A.13] and Proposition B.1.

Now use

$$A[t^{-1}] \simeq \mathrm{colim}(A \xrightarrow{t} A \xrightarrow{t} \cdots).$$

Tensoring with K computes $\mathbf{R}j_*^{\mathrm{Alg}} j^{\mathrm{Alg},*} K$. In the chosen trivialization, the maps $K \otimes \mathcal{L}^n \rightarrow K \otimes \mathcal{L}^{n+1}$ are multiplication by t . A change of trivialization multiplies t by a unit; both sequences then describe the same localization at s . The local formulas therefore glue to the stated equivalence. $\square$

Lemma 4.6. *Let Z be a separated finitely presented X -scheme, let $D \subset Z$ be an effective Cartier divisor, and let $j : U = Z \setminus D \hookrightarrow Z$. Put $\mathcal{I} = \mathcal{O}_Z(-D)$ and $\mathcal{L} = \mathcal{I}^{-1} = \mathcal{O}_Z(D)$. Let $s : \mathcal{O}_Z \rightarrow \mathcal{L}$ be the canonical section, and use the same symbols $\mathcal{L}$ and s for their images under the relative comparison functor. The algebraic extension-of-scalars functor $j^{\mathrm{Alg},*} : \mathcal{C}_{\square}(Z_{\mathcal{X}}^{\mathrm{Alg}}) \rightarrow \mathcal{C}_{\square}(U_{\mathcal{X}}^{\mathrm{Alg}})$ admits a right adjoint $\mathbf{R}j_*^{\mathrm{Alg}}$. For every $K \in \mathcal{C}_{\square}(Z_{\mathcal{X}}^{\mathrm{Alg}})$ there is a natural equivalence*

$$\mathbf{R}j_*^{\mathrm{Alg}} j^{\mathrm{Alg},*} K \simeq \mathrm{colim}_{n \geq 0} (K \otimes \mathcal{L}^{\otimes n}),$$

where the transition maps are multiplication by s . The localization is computed in the algebraic-realization module categories.

Proof. Choose a finite affine cover (Z_a) , subordinate to affine opens of X , on which D is principal. On an intersection $\mathrm{Spec} A$, write $\mathcal{O}(-D) = (t)$. Lemma 3.6 identifies restriction to U with tensoring by $A[t^{-1}]$, whose right adjoint is restriction of scalars.

For $A \rightarrow A'$ on a further intersection, the right-adjoint mate is the equivalence

$$A' \otimes_A^{\mathbf{L}} \mathrm{Res}_{A[t^{-1}]/A} M \simeq \mathrm{Res}_{A'[t^{-1}]/A'} (A'[t^{-1}] \otimes_{A[t^{-1}]}^{\mathbf{L}} M).$$

It follows from $A'[t^{-1}] \simeq A' \otimes_A^{\mathbf{L}} A[t^{-1}]$. Tensor associativity gives its composition compatibility, so Lemmas 3.4 and B.4 glue these right adjoints.

On this affine intersection, we compute

$$\begin{aligned} \mathbf{R}j_*^{\mathrm{Alg}} j^{\mathrm{Alg},*} K &\simeq K \otimes_A^{\mathbf{L}} A[t^{-1}] \\ &\simeq \mathrm{colim}(K \xrightarrow{t} K \xrightarrow{t} \cdots). \end{aligned}$$

The generator t^{-1} trivializes $\mathcal{O}(D)$, and the canonical section is multiplication by t in this trivialization. Thus the last line is $\mathrm{colim}_n (K \otimes \mathcal{O}(nD))$. Changes of local equation give the same localization at s , so these formulas agree on intersections and glue. This is the boundary localization used in [LH26, proof of Theorem A.18]. $\square$

We next choose a modification on which the open is the nonvanishing locus of a section. Over a noetherian scheme, the section comes from an effective Cartier divisor. The second lemma retains the line bundle and section under arbitrary base change.

Lemma 4.7. *Let Y be noetherian and let $j : U \hookrightarrow Y$ be a quasi-compact open immersion. There is a coherent ideal $\mathcal{J} \subset \mathcal{O}_Y$ with vanishing locus $Y \setminus U$ such that, for $p : \tilde{Y} = \text{Bl}_{\mathcal{J}}(Y) \rightarrow Y$, the following hold:*

- (i) *p is projective and of finite presentation (hence proper), and is an isomorphism over U ;*
- (ii) *$\mathcal{J}\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-D)$ is an invertible ideal;*
- (iii) *the complement of U in $\tilde{Y}$ is the effective Cartier divisor D .*

If $Y \rightarrow X$ is proper and of finite presentation, then $\tilde{Y} \rightarrow X$ is proper and of finite presentation.

Proof. Choose a coherent ideal $\mathcal{J}$ with vanishing locus $Y \setminus U$. The blowup of $\mathcal{J}$ is projective and finitely presented, is an isomorphism where $\mathcal{J}$ is the unit ideal, and makes $\mathcal{J}\mathcal{O}_{\tilde{Y}}$ an invertible ideal [Stacks, Section 31.33, especially Lemma 31.33.4]. Its zero locus is precisely $\tilde{Y} \setminus U$. It therefore defines the required effective Cartier divisor. The last assertion follows by composition. $\square$

Lemma 4.8. *Let R be a ring, let S be a proper finitely presented R -scheme, and let $j : U \hookrightarrow S$ be a quasi-compact open immersion. There exist a proper finitely presented morphism $p : \tilde{S} \rightarrow S$ that is an isomorphism over U , a line bundle $\mathcal{L}$ on $\tilde{S}$, and a section $s : \mathcal{O}_{\tilde{S}} \rightarrow \mathcal{L}$ whose nonvanishing locus is the copy of U in $\tilde{S}$.*

Proof. Choose a finite type quasicohherent ideal $\mathcal{J}$ cutting out $S \setminus U$ [Stacks, Tag 01PH]. The approximation used in [Stacks, Tag 0EVA] writes $(S, V(\mathcal{J}))$ as the base change of a pair over a noetherian scheme. Blow up the center on that model, and pull back the resulting modification to S . This gives a proper finitely presented $p : \tilde{S} \rightarrow S$ that is an isomorphism over U .

On the model, the exceptional divisor D_0 is effective Cartier [Stacks, Tag 02OS]. Pull back $\mathcal{O}(D_0)$ and its canonical section to obtain $\mathcal{L}$ and s . Nonvanishing loci commute with base change, so the nonvanishing locus is $p^{-1}(U) \simeq U$. These are exactly the data used in Lemma 4.5. $\square$

4.3. Comparison with Huber analytification. Recall from [LH26, A.19] that the assignment $\text{Spa}(A, A^+) \mapsto \text{AnSpec}(A, A^+)^{\square}$ from Tate affinoid adic spaces to analytic stacks sends rational covers to steady-localization covers and therefore glues to a functor $(-)^{\square} : \{\text{analytic adic spaces}\} \rightarrow \{\text{analytic stacks}\}$.

Lemma 4.9. *Let Z be a quasiseparated analytic adic space and let $(Z_i)_{i \in I}$ be an arbitrary open cover by analytic open subspaces. Put $Z_{i_0 \dots i_n} = Z_{i_0} \cap \dots \cap Z_{i_n}$. Then the canonical morphism of analytic stacks*

$$\left[[n] \mapsto \coprod_{(i_0, \dots, i_n) \in I^{n+1}} Z_{i_0 \dots i_n}^{\square} \right] \longrightarrow Z^{\square}$$

is an equivalence. In particular, if $Z_0 \subset Z_1 \subset \cdots \subset Z$ is an increasing open cover by affinoid subspaces and every $Z_n \hookrightarrow Z_{n+1}$ is a rational open immersion, then

$$\operatorname{colim}_{n \geq 0} Z_n^\square \xrightarrow{\sim} Z^\square.$$

Moreover, every quasicompact affinoid open $W \subset Z$ is contained in some Z_N and admits a finite affinoid cover whose members are rational subdomains of both W and Z_N .

Proof. The functor $(-)^{\square}$ is defined by gluing the affinoid assignment along rational covers [LH26, A.19]. Choose rational-affinoid covers of the Z_i , and refine their intersections by rational affinoids. Quasiseparatedness gives finite refinements on each quasi-compact intersection.

This produces a bisimplicial gluing diagram. Realizing first along the rational covers gives the Čech diagram displayed in the statement. Realizing first along the open cover gives a rational-affinoid presentation of $Z^\square$. The two realizations commute, proving the first assertion. A common refinement compares the choices.

For the increasing cover, each $Z_n^\square \rightarrow Z_{n+1}^\square$ is a monomorphism, since rational open immersions give open localizations [RC24, Proposition 2.3.2]. Its colimit is the union of these subobjects, which the first assertion identifies with $Z^\square$.

Finally, $(W \cap Z_n)$ is an increasing open cover of the quasi-compact W , so $W \subset Z_N$ for some N . Each point of W has a rational neighborhood in Z_N contained in W . That neighborhood is rational in W as well, by base change. A finite collection covers W . $\square$

Proposition 4.10. *Let $V = \operatorname{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid, and let T be a separated finitely presented R -scheme. There is a canonical equivalence of analytic stacks over V*

$$\vartheta_{T,V} : T^{\operatorname{An}} \xrightarrow{\sim} (T^{\operatorname{ad}/V})^\square.$$

It is functorial in T , compatible with Zariski restriction, and compatible with rational base change on V , including the identity, composition, and higher associativity coherences.

Under $\vartheta_{T,V}$, the subcategory $\mathbf{D}_{\operatorname{pc}}^\square(T^{\operatorname{An}})$ defined in [LH26, A.15] is identified with the full subcategory of $\mathcal{C}_\square((T^{\operatorname{ad}/V})^\square)$ whose restriction to every strongly noetherian affinoid open $W = \operatorname{Spa}(C, C^+)$ belongs to the essential image of $\operatorname{Cond}_C : \mathbf{D}_{\operatorname{pc}}(C) \hookrightarrow \mathbf{D}_\square(C, C^+)$. Moreover, if $W \subset c_{T/V}^{-1}(U)$ with $U = \operatorname{Spec} A \subset T$ affine, then for every $M \in \mathbf{D}_{\operatorname{qc}}(T)$ the analytification of M satisfies

$$\alpha_T^{\operatorname{An}}(M)|_{W^\square} \simeq \operatorname{Cond}_C(C \otimes_A^{\mathbf{L}} \mathbf{R}\Gamma(U, M|_U)).$$

Here $\alpha_T^{\operatorname{An}} = j_T^* \alpha_T^{\operatorname{Alg}}$, where $j_T : T^{\operatorname{An}} \rightarrow T^{\operatorname{Alg}}$ is the analytic-locus morphism.

Proof. We compare the bounded charts defining the two realizations, following [LH26, proof of Theorem A.19]. Suppose first that $T = \operatorname{Spec} B$. Choose

$$B = R[T_1, \dots, T_m]/I$$

and a pseudo-uniformizer $\varpi \in R^+$. Let b_i be the image of T_i . Huber analytification is the increasing union of the affinoids

$$W_n = \{ |\varpi^n b_i| \leq 1 \text{ for } 1 \leq i \leq m \}$$

[Hub94, Proposition 3.8] [Zav25, Examples 6.4–6.5]. The inclusion $W_n \subset W_{n+1}$ is rational, given there by $|\varpi^{n+1} b_i| \leq |\varpi|$.

Put

$$\begin{aligned} C_{0,n} &= R\langle \varpi^n T_1, \dots, \varpi^n T_m \rangle, \\ C_n &= C_{0,n}/IC_{0,n}. \end{aligned}$$

Let $C_{0,n}^+ = R^+\langle \varpi^n T_1, \dots, \varpi^n T_m \rangle$, and let C_n^+ be the integral closure of its image in C_n . In the bounded coordinates $U_i = \varpi^n T_i$, the equations are

$$I_n = (f(\varpi^{-n} U_1, \dots, \varpi^{-n} U_m) : f \in I),$$

so $C_n = R\langle \mathbf{U} \rangle / I_n R\langle \mathbf{U} \rangle$. The change of variables identifies the polynomial presentations, and the transition map sends $U_i^{(n+1)}$ to $\varpi U_i^{(n)}$.

We first compute the analytic ring of W_n . Set $P = R[T_1, \dots, T_m]$. The map $P \rightarrow C_{0,n}$ is flat: for a finitely generated ideal $J \subset P$, pull back $\tilde{J} \hookrightarrow \mathcal{O}$ to the closed polydisc. Flatness on local rings [Zav25, Lemma 6.7] and the finite-module theorem [Zav25, Theorem 4.2] give $J \otimes_P C_{0,n} \hookrightarrow C_{0,n}$. Apply the ideal criterion [Stacks, Tag 00HD]. Hence

$$C_{0,n} \otimes_P^{\mathbf{L}} B \simeq C_n[0]$$

in the ordinary derived category.

The ideal $IC_{0,n}$ is closed, and C_n is complete with its quotient topology, by the open mapping theorem and completeness of finite modules [KL19, Theorem 1.2.9 and Corollaries 1.2.11–1.2.12]. Thus $W_n = \mathrm{Spa}(C_n, C_n^+)$.

Let B_n^{bd} be the first, underlined bounded analytic ring of [LH26, A.15]. For the polynomial algebra it is $(C_{0,n}, C_{0,n}^+)^{\square}$ [LH26, Lemma A.8]. For the quotient, the pushout map has the induced analytic structure described in [LH26, proof of Proposition A.16] and [Sch26a, Proposition 12.8]. Before analytic completion its underlying condensed ring is

$$E_n = C_{0,n} \otimes_P^{\mathbf{L}} B,$$

with P and B discrete [Sch26a, Proposition 12.12].

Choose a bounded-above finite-free P -resolution of B . The discrete embedding is exact [And21, Lemma 5.7], and the resulting complex of free condensed P -modules is K-flat. After tensoring with $C_{0,n}$, it is termwise the condensification of the ordinary tensor product. The ordinary calculation above, exactness of condensification, and its agreement with the finite-module topology give

$$E_n \xrightarrow{\sim} C_n[0]$$

as animated condensed rings [And21, Theorem 5.9 and Lemma 5.23].

We next identify completeness. A condensed C_n -complex is complete for this induced structure exactly when its restriction to $C_{0,n}$ is complete. Proposition 3.29 of [And21] expresses that condition using the discrete plus ring $C_{0,n}^+$. Factor

$$C_{0,n}^+ \twoheadrightarrow D_n^+ := \mathrm{im}(C_{0,n}^+ \rightarrow C_n) \longrightarrow C_n^+.$$

Both maps are integral. The restriction criterion and integral invariance [And21, Propositions 3.21–3.22] therefore identify the condition with completeness over $(C_n, C_n^+)^{\square}$. The unit C_n is complete [And21, Lemma 3.24]. Uniqueness of the complete analytic structure [Sch26a, Proposition 12.20] now gives

$$\begin{aligned} B_n^{\mathrm{bd}} &\simeq (C_n, C_n^+)^{\square}, \\ \mathrm{AnSpec} B_n^{\mathrm{bd}} &\simeq W_n^{\square}. \end{aligned}$$

The maps agree with the transitions in n , since both come from the identity on B and enlargement of the coordinate bounds.

Write $\underline{\mathcal{U}}_n = \text{AnSpec } B_n^{\text{bd}}$, and let $\mathcal{U}_n$ denote the second bounded charts in [LH26, A.15]. The factorization of [LH26, A.11], applied to the generators, gives

$$\underline{\mathcal{U}}_n \longrightarrow \mathcal{U}_n \longrightarrow \underline{\mathcal{U}}_{n+1}.$$

The composites are the transitions of the two sequences. Thus the induced maps on colimits are inverse, and Lemma 4.9 gives

$$T^{\text{An}} \simeq \text{colim}_n \underline{\mathcal{U}}_n \simeq \text{colim}_n W_n^\square \simeq (T^{\text{ad}/V})^\square.$$

This defines $\vartheta_{T,V}$ for affine T .

Pullback along these chart maps preserves nuclearity. Indeed, scalar extension takes each measure generator to its counterpart and hence preserves compact objects. It is strong symmetric monoidal and preserves colimits. The natural map from the image of a dual to the dual of the image takes trace-class maps to trace-class maps. It therefore takes the presentations of basic nuclear objects, and then their colimits, to nuclear objects [And21, Propositions 2.9 and 2.11, Definition 5.32, and Proposition 5.33] [CS26, Definition 8.5 and Theorem 8.6].

Now take $W = \text{Spa}(C, C^+) \subset c_{T/V}^{-1}(U)$ with $U = \text{Spec } A$, and put $K = \mathbf{R}\Gamma(U, M|_U)$. Lemma 4.9 puts W in some W_N and gives a finite cover by $W_j = \text{Spa}(C_j, C_j^+)$ rational in both. On W_j , scalar extension and Zariski restriction give

$$\alpha_T^{\text{An}}(M)|_{W_j^\square} \simeq \text{Cond}_{C_j}(C_j \otimes_A^{\mathbf{L}} K)$$

[And21, Lemma 5.10]. This is also the restriction of $\text{Cond}_C(C \otimes_A^{\mathbf{L}} K)$, since

$$C_j \otimes_C^{\mathbf{L}} (C \otimes_A^{\mathbf{L}} K) \simeq C_j \otimes_A^{\mathbf{L}} K.$$

Rational descent proves the stated formula on W . The common bounded refinements of Lemma B.9 give independence of the presentation and compatibility with rational base change. On those refinements, the maps are the same scalar-extension maps, with their tensor-associativity coherences.

We finish the affine case by comparing the finiteness conditions. For $K \in \mathbf{D}_{\text{pc}}^\square(T^{\text{An}})$, its restriction to $\mathcal{U}_N$ is nuclear, so its restriction to $\underline{\mathcal{U}}_N = W_N^\square$ is nuclear as well. Pseudocompactness on this underlined chart is part of the definition. Cover an arbitrary affinoid $W \subset W_N$ by common rational subdomains as above. Nuclearity descends to $W^\square$ by [And21, Theorem 5.42]. These rational maps are open localizations [RC24, Proposition 2.3.2], so pseudocompactness descends by [LH26, Proposition A.5]. The affinoid identification in [LH26, A.15] then puts $K|_{W^\square}$ in $\text{Cond}_C(\mathbf{D}_{\text{pc}}(C))$.

Conversely, suppose the restriction to each Huber affinoid is in that image. On every $\underline{\mathcal{U}}_n = W_n^\square$, the object is nuclear and pseudocompact. Descent over the underlined charts gives pseudocompactness on T^{An} [LH26, A.15 and Proposition A.5]. Pulling back from $\underline{\mathcal{U}}_{n+1}$ to $\mathcal{U}_n$ gives nuclearity on the second system, and its descent gives global nuclearity [LH26, A.15 and Proposition A.4]. This proves equality of the subcategories; compare also [And21, Theorems 5.42, 5.44, and 5.50].

For separated T , choose a finite affine Zariski cover. The affine comparisons agree on all intersections. Huber analytification glues these opens, and Lemma 4.9 identifies its associated analytic stack with their geometric realization. The same realization defines T^{An}

[LH26, Theorem A.13]. Gluing gives $\vartheta_{T,V}$, the affinoid object formula, and the equality of finiteness subcategories. The affine functoriality and base-change compatibilities descend with them. $\square$

Lemma 4.11. *Let (R, R^+) be a strongly noetherian complete Tate Huber pair, let $h : S' \rightarrow S$ be a morphism of separated finitely presented R -schemes, let $K \in \mathbf{D}_{\text{pc}}^\square(S^{\text{An}})$, and let L be a line bundle on S' . Then $h^{\text{An},*}K \otimes L^{\text{An}} \in \mathbf{D}_{\text{pc}}^\square((S')^{\text{An}})$. The construction is functorial in h and compatible with composition.*

Proof. Use Proposition 4.10. Cover a strongly noetherian affinoid $W' \subset (S')^{\text{ad}/V}$ by finitely many rational affinoids

$$W'_\nu = \text{Spa}(C'_\nu, C'^+_\nu)$$

that map into strongly noetherian affinoids $W_\nu = \text{Spa}(C_\nu, C^+_\nu)$ of $S^{\text{ad}/V}$. Refine so that L is trivial on every W'_ν .

Write $K|_{W'_\nu} \simeq \text{Cond}_{C'_\nu}(P_\nu)$ with $P_\nu \in \mathbf{D}_{\text{pc}}(C'_\nu)$. The pullback formula is

$$h^{\text{An},*}K|_{(W'_\nu)^\square} \simeq \text{Cond}_{C'_\nu}(C'_\nu \otimes_{C'_\nu}^{\mathbf{L}} P_\nu).$$

A bounded-above finite-free resolution stays such after scalar extension. Thus each right side is pseudocoherent, and tensoring with the trivialization of L preserves it. Rational descent proves the assertion on W' , and then on $(S')^{\text{An}}$. The maps are the scalar-extension maps of [And21, Lemma 5.10], so tensor associativity gives composition compatibility. $\square$

Lemma 4.12. *Let Z and Z' be proper and finitely presented over X , let $h : Z' \rightarrow Z$ be an X -morphism, let $K \in \mathbf{D}_{\text{pc}}^\square(Z^{\text{An}}_{\mathcal{X}})$, and let L be a line bundle on Z' . Then $h^{\text{An},*}K \otimes L^{\text{An}} \in \mathbf{D}_{\text{pc}}^\square((Z')^{\text{An}}_{\mathcal{X}})$. Here L^{An} denotes the corresponding invertible object on the relative analytic realization.*

Proof. Over $V = \text{Spa}(R, R^+) \subset \mathcal{X}$, the object in the statement restricts to

$$h_R^{\text{An},*}(K|_{Z^{\text{An}}_R}) \otimes L_R^{\text{An}},$$

where $Z_R = Z \times_X \text{Spec } R$ and $Z'_R = Z' \times_X \text{Spec } R$. It is pseudocoherent by Lemma 4.11. Proposition 3.3 tests membership on these base charts and gives the result. $\square$

Lemma 4.13. *Let $V = \text{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid, and let T be a proper finitely presented R -scheme. There are a finite cover by strongly noetherian affinoid opens $T^{\text{ad}/V} = \bigcup_{i=1}^r W_i$, with $W_i = \text{Spa}(B_i, B_i^+)$, and affine opens $U_i = \text{Spec } A_i \subset T$ with $W_i \subset c_{T/V}^{-1}(U_i)$ such that*

$$\coprod_{i=1}^r \text{Spec } B_i \longrightarrow T$$

is fpqc.

Proof. The space $T^{\text{ad}/V}$ is proper over V and hence quasi-compact [Zav25, Remark 6.3]. Choose a finite affinoid cover (W_i) subordinate to the inverse images of an affine cover of T , and write $W_i = \text{Spa}(B_i, B_i^+)$ and $U_i = \text{Spec } A_i$.

For flatness, take a finitely generated ideal $J \subset A_i$. Flatness of $c_{T/V}$ on local rings [Zav25, Lemma 6.7] gives

$$c_{T/V}^* \tilde{J}|_{W_i} \hookrightarrow \mathcal{O}_{W_i}.$$

The ring A_i is noetherian, so J has a finite presentation. Pulling it back identifies the source with the sheaf associated to $J \otimes_{A_i} B_i$. The finite-module equivalence [Zav25, Theorem 4.2] therefore gives $J \otimes_{A_i} B_i \hookrightarrow B_i$. The ideal criterion makes $A_i \rightarrow B_i$ flat [Stacks, Tag 00HD].

Let x be a closed point of T . By Proposition B.11 and properness, the nonzero skyscraper $\kappa(x)$ has nonzero image under α_T^{An} . It is nonzero on some W_i . Proposition 4.10 identifies that restriction with

$$\text{Cond}_{B_i}(B_i \otimes_{A_i}^{\mathbf{L}} \Gamma(U_i, \kappa(x)|_{U_i})).$$

Thus $x \in U_i$ and $B_i \otimes_{A_i} \kappa(x) \neq 0$. A prime of this ring gives a point above x .

Every point $y \in T$ specializes to a closed point x . Since U_i is open, $x \in U_i$ implies $y \in U_i$. Flat going-down lifts $\mathfrak{p}_y \subset \mathfrak{p}_x$ to a prime below a chosen prime over x . Hence the family covers every point of T . It is a finite family of affine flat maps, so its disjoint union is fpqc. $\square$

Proposition 4.14. *Let (R, R^+) be a strongly noetherian complete Tate Huber pair and let S be a proper scheme of finite presentation over R . Analytification induces an equivalence*

$$\alpha_S^{\text{An}} : \text{D}_{\text{pc}}(S) \xrightarrow{\sim} \text{D}_{\text{pc}}^{\square}(S^{\text{An}}).$$

This is the strongly noetherian case of [LH26, Theorem A.18].

Proof. Analytification preserves pseudocoherence by the bounded-chart calculation in [LH26, A.15 and the paragraph preceding Theorem A.18]. Properness gives $S^{\text{An}} \simeq S^{\text{Alg}}$ [LH26, Theorem A.13], and Proposition B.11 gives full faithfulness.

Take $C \in \text{D}_{\text{pc}}^{\square}(S^{\text{An}})$ and an affine open $j : U = \text{Spec } A \hookrightarrow S$. By Lemma 4.8, choose a proper finitely presented modification $p : \tilde{S} \rightarrow S$, equal to the identity over U , and a section of a line bundle $\mathcal{L}$ whose nonvanishing locus is U . Write $\tilde{j} : U \hookrightarrow \tilde{S}$, $\tilde{\pi} : \tilde{S} \rightarrow \text{Spec } R$, and $\tilde{C} = p^{\text{An},*}C$. Properness identifies the two realizations of $\tilde{S}$.

By Lemma 4.11, each $\tilde{C} \otimes \mathcal{L}^n$ is pseudocoherent. Localization gives

$$\mathbf{R}\tilde{j}_*^{\text{Alg}} \tilde{j}^{\text{Alg},*} \tilde{C} \simeq \text{colim}_n (\tilde{C} \otimes \mathcal{L}^n).$$

Apply $\mathbf{R}\tilde{\pi}_*$. Theorem 4.1 puts

$$E_n = \mathbf{R}\tilde{\pi}_*(\tilde{C} \otimes \mathcal{L}^n)$$

in $\text{Cond}_R(\text{D}_{\text{pc}}(R))$, and Corollary 4.2 allows the colimit through direct image. By full faithfulness and Lemma B.2, lift $E_{\bullet}$ to $M_{\bullet}$ in $\text{D}_{\text{pc}}(R)$ and put $M = \text{colim}_n M_n$. We get

$$\mathbf{R}\tilde{\pi}_* \mathbf{R}\tilde{j}_*^{\text{Alg}} \tilde{j}^{\text{Alg},*} \tilde{C} \simeq \text{Cond}_R(M).$$

Since $\tilde{\pi} \circ \tilde{j} = q : U \rightarrow \text{Spec } R$, the left side is $\mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*} C$: the two direct-image functors are right adjoint to the same pullback. Lemmas B.10 and B.3 lift its A -action and give $N_U \in \text{D}(A)$ with

$$\alpha_A^{\text{Alg}}(N_U) \simeq j^{\text{Alg},*} C.$$

Do this on a finite affine cover of S . On intersections, full faithfulness lifts the identifications with the restriction of C , together with their cocycle homotopies. Zariski descent then gives $N \in \text{D}_{\text{qc}}(S)$ and $\alpha_S^{\text{An}}(N) \simeq C$.

To check finiteness, use the fpqc family $s_i : \text{Spec } B_i \rightarrow S$ from Lemma 4.13. Put $N_i = \mathbf{L}s_i^* N$. The affinoid formula in Proposition 4.10 gives

$$\text{Cond}_{B_i}(N_i) \simeq C|_{W_i^{\square}} \simeq \text{Cond}_{B_i}(P_i)$$

for some $P_i \in D_{\text{pc}}(B_i)$. Full faithfulness identifies N_i with P_i . Faithfully flat descent for pseudocoherence [Stacks, Tags 066D and 068R] gives $N \in D_{\text{pc}}(S)$, completing the proof. $\square$

5. GLOBAL PSEUDOCOHERENT GAGA

We can now prove GAGA. The boundary formula will give an algebraic complex on each affine open, and the following fpqc test will show that it is pseudocoherent. Full faithfulness then glues these complexes.

5.1. Finite fpqc reflection of pseudocoherence.

Proposition 5.1. *Let $f : Y \rightarrow X$ be proper and of finite presentation, let $j : U \hookrightarrow Y$ be an affine open for which the composite $q : U \rightarrow X$ is affine, and identify Y_X^{An} with Y_X^{Alg} by properness. Let $C \in D_{\text{pc}}^\square(Y_X^{\text{An}})$, and suppose that $N \in D_{\text{qc}}(U)$ is equipped with an equivalence $\alpha_U(N) \simeq j^{\text{Alg},*}C$ in $\mathcal{C}_\square(U_X^{\text{Alg}})$. Then $N \in D_{\text{pc}}(U)$.*

Proof. Choose the fpqc family of Lemma 2.7 and form

$$\begin{aligned} Y_i &= Y \times_X \text{Spec } R_i, \\ U_i &= U \times_X \text{Spec } R_i = \text{Spec } B_i. \end{aligned}$$

Write $U = \text{Spec } B$. Since the family is finite, its base change to U is the faithfully flat map

$$B \longrightarrow B' = \prod_i B_i.$$

We will check pseudocoherence of $N_i = N \otimes_B^{\mathbf{L}} B_i$.

Let C_i be the restriction of C to Y_i^{An} . The scheme Y_i is proper over R_i , so Proposition 4.14 gives $M_i \in D_{\text{pc}}(Y_i)$ realizing C_i . Restrict the given equivalence $\alpha_U(N) \simeq j^{\text{Alg},*}C$ to V_i . Over (R_i, R_i^+) , this identifies the images of N_i and $M_i|_{U_i}$ under the comparison functor. Proposition B.11 therefore gives

$$N_i \simeq M_i|_{U_i} \in D_{\text{pc}}(B_i).$$

The finite product $N \otimes_B^{\mathbf{L}} B' \simeq \prod_i N_i$ is pseudocoherent over B' . Faithfully flat descent [Stacks, Tag 068R] gives $N \in D_{\text{pc}}(B)$. $\square$

5.2. Boundary pushforward and affine algebraization. We now make the main calculation. It is the boundary-localization argument of [CS26, Theorem 13.10] and [LH26, Theorem A.18], followed by the comparison on the base curve.

Theorem 5.2. *Let $f : Y \rightarrow X$ be proper and of finite presentation, and let $j : U \hookrightarrow Y$ be an affine open for which $q = f \circ j : U \rightarrow X$ is affine. Choose a boundary blowup $p : \tilde{Y} \rightarrow Y$ as in Lemma 4.7, identify $p^{-1}(U)$ with U , and let D be the resulting effective Cartier divisor with complement U . Write $\tilde{f} = f \circ p$, and use properness to identify the algebraic and analytic realizations of Y and $\tilde{Y}$.*

For $C \in D_{\text{pc}}^\square(Y_X^{\text{An}})$, put $\tilde{C} = (p^{\text{An}})^*C$ and, for $n \geq 0$,

$$E_n(C) = \mathbf{R}\tilde{f}_*^{\text{An}}(\tilde{C} \otimes \mathcal{O}_{\tilde{Y}}(nD)),$$

where the line bundles denote their relative realizations. The canonical section $\mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\tilde{Y}}(D)$ defines the transition maps $E_n(C) \rightarrow E_{n+1}(C)$.

There is a canonical equivalence, natural in C ,

$$\mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*} C \simeq \text{colim}_{n \geq 0} \mathbf{R}\tilde{f}_*^{\text{An}} (\tilde{C} \otimes \mathcal{O}_{\tilde{Y}}(nD))$$

in $\mathcal{C}_{\square}(\mathcal{X})$.

Every $E_n(C)$ belongs to $\mathbf{D}_{\text{pc}}^{\square}(\mathcal{X})$. The whole sequential diagram has an algebraic lift $M_{\bullet}(C)$ in $\mathbf{D}_{\text{pc}}(X)$, functorial in C , with an equivalence of diagrams $\alpha_X(M_{\bullet}(C)) \simeq E_{\bullet}(C)$. For

$$M_U(C) = \text{colim}_{n \geq 0} M_n(C) \in \mathbf{D}_{\text{qc}}(X),$$

there is consequently a natural equivalence

$$\alpha_X(M_U(C)) \simeq \mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*} C.$$

The lift of the diagram, equipped with its comparison, is unique up to contractible choice. The algebraic object $M_U(C)$, equipped with the last equivalence, is independent of the boundary blowup up to canonical equivalence.

Proof. Let $\tilde{j} : U \hookrightarrow \tilde{Y}$ be the open immersion. The scheme $\tilde{Y}$ is proper over X , so Proposition 3.2 identifies its algebraic and analytic realizations. By Lemma 4.12, each $\tilde{C} \otimes \mathcal{O}_{\tilde{Y}}(nD)$ is pseudocoherent. Proposition 4.4 therefore gives $E_n(C) \in \mathbf{D}_{\text{pc}}^{\square}(\mathcal{X})$.

Cartier localization gives

$$\mathbf{R}\tilde{j}_*^{\text{Alg}} \tilde{j}^{\text{Alg},*} \tilde{C} \simeq \text{colim}_n (\tilde{C} \otimes \mathcal{O}_{\tilde{Y}}(nD)).$$

Since p is the identity over U , we have $\tilde{j}^{\text{Alg},*} \tilde{C} \simeq j^{\text{Alg},*} C$. Moreover, $\tilde{f} \circ \tilde{j} = q$, so $\mathbf{R}\tilde{f}_*^{\text{An}} \mathbf{R}\tilde{j}_*^{\text{Alg}}$ is right adjoint to $q^{\text{Alg},*}$. It is therefore $\mathbf{R}q_*^{\text{Alg}}$. Using the colimit assertion of Proposition 4.4, we compute

$$\begin{aligned} \mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*} C &\simeq \mathbf{R}\tilde{f}_*^{\text{An}} \mathbf{R}\tilde{j}_*^{\text{Alg}} \tilde{j}^{\text{Alg},*} \tilde{C} \\ &\simeq \mathbf{R}\tilde{f}_*^{\text{An}} \left(\text{colim}_n \tilde{C} \otimes \mathcal{O}_{\tilde{Y}}(nD) \right) \\ &\simeq \text{colim}_n E_n(C). \end{aligned}$$

The transitions are induced by the section of $\mathcal{O}(D)$, so this is the required formula, naturally in C .

Apply the inverse of the base equivalence in Proposition 2.10 to the diagram $E_{\bullet}(C)$. Lemma B.2 gives its lift $M_{\bullet}(C)$, together with the comparison, uniquely up to contractible choice. Taking the colimit gives

$$\alpha_X \left(\text{colim}_n M_n(C) \right) \simeq \text{colim}_n E_n(C) \simeq \mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*} C.$$

All constructions are functorial in C . A different blowup gives another preimage of the same object on the right. Full faithfulness of α_X identifies these preimages with their comparisons, proving the final assertion. $\square$

The theorem gives an object of $\mathbf{D}_{\text{qc}}(X)$ with an action of the affine coordinate algebra. Lifting that action gives the complex on U to which Proposition 5.1 applies.

Proposition 5.3. *Let $f : Y \rightarrow X$ be proper and of finite presentation, let $C \in \mathbf{D}_{\text{pc}}^{\square}(Y_{\mathcal{X}}^{\text{An}})$, and let $j : U = \text{Spec } B \hookrightarrow Y$ be an affine open contained in $f^{-1}(W)$ for a standard affine*

$W \subset X$. Then there is an object $N_U \in D_{\text{pc}}(U)$ and an equivalence $\alpha_U(N_U) \simeq j^{\text{Alg},*}C$ in $\mathcal{C}_{\square}(U_{\mathcal{X}}^{\text{Alg}})$.

Proof. The map $q : U \rightarrow W \rightarrow X$ is affine because $W \hookrightarrow X$ is affine. Identify $Y_{\mathcal{X}}^{\text{An}}$ with $Y_{\mathcal{X}}^{\text{Alg}}$ using properness. Theorem 5.2 gives $M \in D_{\text{qc}}(X)$ and

$$\mathbf{R}q_*^{\text{Alg}} j^{\text{Alg},*}C \simeq \alpha_X(M).$$

Put $\mathcal{A} = q_*\mathcal{O}_U$. By Lemma 3.6, the object $j^{\text{Alg},*}C$ is an $\alpha_X(\mathcal{A})$ -module with underlying object $\alpha_X(M)$. Its action is a map

$$\alpha_X(\mathcal{A} \otimes M) \longrightarrow \alpha_X(M).$$

Full faithfulness lifts this map to $\mathcal{A} \otimes M \rightarrow M$, and Lemma B.3 lifts the full module structure. Thus we obtain $N_U \in \text{Mod}_{\mathcal{A}}(D_{\text{qc}}(X)) \simeq D_{\text{qc}}(U)$ with

$$\alpha_U(N_U) \simeq j^{\text{Alg},*}C.$$

Proposition 5.1 proves that N_U is pseudocoherent. $\square$

5.3. The stable equivalence and proper pushforward. We can now finish the proof by gluing the affine algebraizations. To include proper direct images, we will use the following algebraic finiteness statement.

Lemma 5.4. *Let $h : S \rightarrow T$ be a proper morphism of finite presentation between noetherian schemes. Then $\mathbf{R}h_* : D_{\text{coh}}^-(S) \rightarrow D_{\text{coh}}^-(T)$.*

Proof. Choose a finite affine cover (T_a) of T . Cover each $S_a = h^{-1}(T_a)$ by r_a affine opens. Their intersections are affine because h is separated. The alternating Čech complex therefore gives cohomological dimension at most $r_a - 1$ on S_a . This cover stays affine after restriction to any affine open of T_a , so it gives the same bound for direct image. With $d = \max_a(r_a - 1)$, the finite Čech calculation gives

$$\mathbf{R}h_*(D_{\text{qc}}(S)^{\leq q}) \subset D_{\text{qc}}(T)^{\leq q+d}.$$

Take $M \in D_{\text{coh}}^-(S)^{\leq N}$. Its direct image is bounded above by $N + d$. For a fixed n , apply $\mathbf{R}h_*$ to

$$\tau_{\leq n-d-1}M \longrightarrow M \longrightarrow \tau_{\geq n-d}M.$$

The first direct image lies in $D^{\leq n-1}$, so

$$H^n(\mathbf{R}h_*M) \simeq H^n(\mathbf{R}h_*\tau_{\geq n-d}M).$$

The truncated complex is bounded coherent. Proper coherence and its finite Postnikov tower make the right side coherent [Stacks, Tags 0BCF and 02O5]. This holds for every n and proves the lemma. $\square$

Theorem 5.5. *Let $f : Y \rightarrow X$ be proper. Relative analytic realization induces an equivalence of stable ∞ -categories*

$$\alpha_Y : D_{\text{pc}}(Y) \xrightarrow{\sim} D_{\text{pc}}^{\square}(Y_{\mathcal{X}}^{\text{An}}).$$

For every $M \in D_{\text{pc}}(Y)$, the canonical comparison morphism

$$\alpha_X(\mathbf{R}f_*M) \longrightarrow \mathbf{R}f_*^{\text{An}}(\alpha_Y M)$$

is an equivalence in $\mathcal{C}_{\square}(\mathcal{X})$. Consequently,

$$\mathbf{R}\Gamma(Y, M) \simeq \mathbf{R}\Gamma(Y_{\mathcal{X}}^{\text{An}}, \alpha_Y M).$$

Proof. Properness identifies $Y_{\mathcal{X}}^{\text{An}}$ with $Y_{\mathcal{X}}^{\text{Alg}}$. Proposition 3.7 gives full faithfulness, and Proposition 3.8 gives preservation of pseudocoherence.

For essential surjectivity, take $C \in \mathbf{D}_{\text{pc}}^{\square}(Y_{\mathcal{X}}^{\text{An}})$ and a finite affine cover (U_a) subordinate to standard affines of X . Proposition 5.3 gives $N_a \in \mathbf{D}_{\text{pc}}(U_a)$ with

$$\alpha_{U_a}(N_a) \simeq C|_{U_a^{\text{Alg}}}.$$

On $U_a \cap U_b$, the two restrictions realize the same object. Full faithfulness lifts their specified equivalence. Lemma B.2 lifts the cocycle homotopies on all higher intersections as well. Thus the N_a form descent data on the full Čech nerve. Pseudocoherent Zariski descent gives $N \in \mathbf{D}_{\text{pc}}(Y)$ and $\alpha_Y(N) \simeq C$.

We next compare proper direct images. Applying α_Y to the algebraic counit gives

$$(f^{\text{An}})^* \alpha_X(\mathbf{R}f_* M) \simeq \alpha_Y(\mathbf{L}f^* \mathbf{R}f_* M) \longrightarrow \alpha_Y(M).$$

Its analytic adjoint is the canonical map

$$\gamma_M : \alpha_X(\mathbf{R}f_* M) \longrightarrow E := \mathbf{R}f_*^{\text{An}} \alpha_Y(M).$$

Lemma 5.4 gives $\mathbf{R}f_* M \in \mathbf{D}_{\text{pc}}(X)$, while Propositions 3.8 and 4.4 give $E \in \mathbf{D}_{\text{pc}}^{\square}(\mathcal{X})$. Write $E \simeq \alpha_X(N)$ using base GAGA. Full faithfulness lifts γ_M to a map $u_M : \mathbf{R}f_* M \rightarrow N$.

For $P \in \text{Perf}(X)$, the two adjunctions give

$$\begin{aligned} \text{Map}_X(P, \mathbf{R}f_* M) &\simeq \text{Map}_Y(f^* P, M), \\ \text{Map}_X(P, N) &\simeq \text{Map}_{\mathcal{X}}(\alpha_X P, E) \\ &\simeq \text{Map}_{Y_{\mathcal{X}}^{\text{An}}}((f^{\text{An}})^* \alpha_X P, \alpha_Y M) \\ &\simeq \text{Map}_{Y_{\mathcal{X}}^{\text{An}}}(\alpha_Y(f^* P), \alpha_Y M) \\ &\simeq \text{Map}_Y(f^* P, M). \end{aligned}$$

The map induced by u_M is the identity under these identifications, because γ_M was obtained from the algebraic counit. Perfect complexes generate $\mathbf{D}_{\text{qc}}(X)$, so u_M is an equivalence. Hence γ_M is an equivalence.

Finally, apply full faithfulness to the unit:

$$\mathbf{R}\Gamma(Y, M) = \text{Map}_Y(\mathcal{O}_Y, M) \simeq \text{Map}_{Y_{\mathcal{X}}^{\text{An}}}(\mathcal{O}, \alpha_Y M).$$

This gives the cohomology comparison. □

6. THE TRANSPORTED COHERENT STRUCTURE AND CLASSICAL GAGA

We now identify the categories in the stable comparison with ordinary coherent sheaves and bounded coherent complexes on Huber analytification. The tensor product on coherent hearts is the usual one, given by H^0 of derived tensor product.

6.1. The transported t -structure. Let $T \rightarrow X$ be proper and of finite presentation, and put $\mathcal{P}_T = \mathbf{D}_{\text{pc}}^{\square}(T_{\mathcal{X}}^{\text{An}})$. By Theorem 5.5, $\alpha_T : \mathbf{D}_{\text{pc}}(T) \rightarrow \mathcal{P}_T$ is an equivalence. We equip $\mathcal{P}_T$ with the t -structure transported from the standard coherent t -structure on $\mathbf{D}_{\text{pc}}(T)$. Write $\mathcal{P}_T^{\heartsuit}$ for its heart and $\mathcal{P}_T^b$ for its full subcategory of objects bounded on both sides.

Lemma 6.1. *Let $V = \text{Spa}(R, R^+) \in \text{RatAff}_X(\mathcal{X})$ and put $T_R = T \times_X \text{Spec } R$. Under rational restriction $\mathcal{P}_T \rightarrow \mathbf{D}_{\text{pc}}^{\square}(T_R^{\text{An}})$, the transported t -structure restricts to the t -structure*

transported from $D_{\text{pc}}(T_R)$ through the equivalence $\alpha_{T_R}^{\text{An}} : D_{\text{pc}}(T_R) \xrightarrow{\sim} D_{\text{pc}}^{\square}(T_R^{\text{An}})$ of Proposition 4.14. The restriction functors for rational localizations are t -exact.

Proof. Put $M_R = \mathbf{L}(T_R \rightarrow T)^* M$. Rational base change gives

$$\rho_V \alpha_T(M) \simeq \alpha_{T_R}^{\text{An}}(M_R).$$

The base ring maps are flat by Lemma 2.4. Thus $M \mapsto M_R$ is t -exact, giving the stated compatibility of transported structures. A rational localization $R \rightarrow R'$ is also flat, so the same argument proves t -exactness of the restriction functors. $\square$

6.2. Bounded and coherent comparison. We begin with open descent for ordinary derived categories. Finite covers provide the uniform bounds needed to compare their bounded coherent parts with the solid categories.

Lemma 6.2. *Let $(Z, \mathcal{O}_Z)$ be a ringed space, let $(Z_i)_{i \in I}$ be an open cover, and fix an integer a . If $Z_{i_0 \dots i_n} = Z_{i_0} \cap \dots \cap Z_{i_n}$, then restriction induces an equivalence of ∞ -categories*

$$D^{\geq a}(\mathcal{O}_Z) \xrightarrow{\sim} \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n) \in I^{n+1}} D^{\geq a}(\mathcal{O}_{Z_{i_0 \dots i_n}}).$$

Thus uniformly left-bounded complexes and their mapping spectra satisfy effective open descent.

Proof. Let $H\mathcal{O}_Z$ be the Eilenberg–MacLane sheaf of rings. Sheaves of spectra satisfy open descent, and forming modules commutes with limits. Hence

$$\text{Mod}_{H\mathcal{O}_Z}(\text{Shv}(Z, \text{Sp}))$$

is computed by the module categories on the Čech nerve. Its standard t -structure is local, so the same is true with a fixed cohomological lower bound a .

For this 1-localic topos and discrete sheaf of rings, [Lur11, Proposition 2.1.8] identifies the left-bounded derived category of $\mathcal{O}_Z$ -modules with the left-bounded objects of the module-spectrum category. Translating its homological indexing to our convention and restricting to degrees $\geq a$ gives the formula. The equivalence also identifies the mapping spectra inherited from the ambient stable categories; see [Lur17, Section 4.8.5]. $\square$

Lemma 6.3. *Let $p : Z \rightarrow S$ be a quasi-compact and quasi-separated morphism of locally noetherian adic spaces, assume that S is quasiseparated, and let $S = V_1 \cup \dots \cup V_r$ be a finite cover by quasicompact open subsets. Write*

$$\begin{aligned} V_{i_0 \dots i_n} &= V_{i_0} \cap \dots \cap V_{i_n}, \\ Z_{i_0 \dots i_n} &= Z \times_S V_{i_0 \dots i_n}. \end{aligned}$$

Then restriction induces an equivalence

$$D_{\text{coh}}^b(Z) \xrightarrow{\sim} \lim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n)} D_{\text{coh}}^b(Z_{i_0 \dots i_n}).$$

The same statement holds for the full subcategories of complexes with cohomological amplitude in a fixed interval $[a, b]$.

Proof. Fix $[a, b]$. By Lemma 6.2, complexes in degrees $\geq a$ and their maps descend. For the descended complex, vanishing above b and coherence of its cohomology can both be checked on the covering opens. This proves descent with fixed amplitude.

For arbitrary bounded coherent descent data, choose one interval $[a, b]$ containing the cohomology of the finitely many complexes on Z_i . All higher components are restrictions of these, so the same interval applies to them. The fixed-amplitude assertion then gives a global bounded coherent complex. Choosing a common lower bound for two such complexes and using mapping descent proves full faithfulness. $\square$

Lemma 6.4. *Let S be a quasicompact quasiseparated space, and let $\mathcal{B}$ be a basis of quasicompact opens such that every finite intersection of members of $\mathcal{B}$ admits a finite cover by members of $\mathcal{B}$. Let $\mathcal{F}$ be a presheaf of ∞ -categories on the quasicompact opens of S that satisfies finite Čech descent, including descent of mapping spaces. Then restriction induces an equivalence*

$$\mathcal{F}(S) \xrightarrow{\sim} \mathrm{Cart}_{V \in \mathcal{B}} \mathcal{F}(V).$$

Proof. Let $j : \mathcal{B} \hookrightarrow \mathrm{Op}_{\mathrm{qc}}(S)$ be the inclusion. For a presheaf $\mathcal{G}$ on $\mathcal{B}$, its right Kan extension is

$$(j_* \mathcal{G})(U) = \lim_{V \in \mathcal{B}, V \subset U} \mathcal{G}(V),$$

with the restriction maps in the diagram.

Every quasi-compact open admits a finite basis cover, and every finite intersection in such a cover admits a finite basis refinement. Finite Čech descent therefore computes $\mathcal{F}(U)$ and $(j_* j^* \mathcal{F})(U)$ from the same local data. Their mapping spaces are computed in the same way. A common refinement compares two choices, giving a natural equivalence

$$\mathcal{F}(U) \xrightarrow{\sim} (j_* j^* \mathcal{F})(U).$$

Evaluate at S . The limit on the right is exactly the category of cartesian sections in the statement. $\square$

Lemma 6.5. *Let $p : Z \rightarrow S$ be a quasi-compact and quasi-separated morphism of locally noetherian adic spaces, with S quasicompact and quasiseparated, and let $\mathcal{B}$ be a rational-affinoid basis of S . For $V \in \mathcal{B}$, put $Z_V = Z \times_S V$. Restriction induces an equivalence*

$$\mathrm{D}_{\mathrm{coh}}^b(Z) \xrightarrow{\sim} \mathrm{Cart}_{V \in \mathcal{B}} (\mathrm{D}_{\mathrm{coh}}^b(Z_V)).$$

Proof. Choose a finite basis cover $S = \bigcup_i V_i$. For a cartesian section (K_V) , choose an interval $[a, b]$ containing the cohomology of all K_{V_i} . For any other $V \in \mathcal{B}$, choose a finite rational refinement

$$V = \bigcup_{i, \nu} W_{i\nu}$$

with $W_{i\nu} \subset V \cap V_i$. Cartesianness identifies K_V and K_{V_i} after restriction to $W_{i\nu}$. Thus K_V also has amplitude in $[a, b]$.

The categories $\mathcal{F}_{[a, b]}(U) = \mathrm{D}_{\mathrm{coh}}^b(Z_U)^{[a, b]}$ satisfy finite descent by Lemma 6.3. Lemma 6.4 therefore descends the given section to an object of $\mathrm{D}_{\mathrm{coh}}^b(Z)$. Every global bounded coherent complex gives such a section, and the mapping-space part of the same descent proves full faithfulness. $\square$

Proposition 6.6. *Let Z be a quasicompact quasiseparated locally strongly noetherian analytic adic space. Let $\mathrm{D}_{\mathrm{pc}}^\square(Z^\square)$ denote the full subcategory of $\mathcal{C}_\square(Z^\square)$ whose restriction to every strongly noetherian affinoid open $W = \mathrm{Spa}(A, A^+)$ belongs to the image of $\mathrm{D}_{\mathrm{pc}}(A) \rightarrow \mathrm{D}_\square(A, A^+)$. Flat rational localization gives this category a chartwise t -structure.*

Condensification induces a canonical exact symmetric monoidal equivalence on hearts and a canonical exact equivalence on bounded objects

$$\begin{aligned} \mathrm{Coh}(Z) &\xrightarrow{\sim} \mathrm{D}_{\mathrm{pc}}^{\square}(Z^{\square})_{\mathrm{ch}}^{\heartsuit}, \\ \mathrm{D}_{\mathrm{coh}}^b(Z) &\xrightarrow{\sim} \mathrm{D}_{\mathrm{pc}}^{\square}(Z^{\square})_{\mathrm{ch}}^b. \end{aligned}$$

These equivalences commute with restriction to analytic affinoid opens.

Proof. First let $W = \mathrm{Spa}(A, A^+)$ be strongly noetherian affinoid. The finite-module theorem gives

$$\mathrm{Mod}_{\mathrm{fg}}(A) \simeq \mathrm{Coh}(W)$$

and coherent acyclicity [Zav25, Theorem 4.2]. Let $\mathrm{D}_{\mathrm{fg}}^b(A)$ denote bounded complexes with finite cohomology. Since A is noetherian, this is the bounded derived category of finite modules, viewed inside $\mathrm{D}(A)$.

Derived sheafification and sections give an adjunction

$$(-)^{\sim} : \mathrm{D}_{\mathrm{fg}}^b(A) \rightleftarrows \mathrm{D}_{\mathrm{coh}}^b(W) : \mathbf{R}\Gamma(W, -).$$

For $K \in \mathrm{D}_{\mathrm{coh}}^b(W)$, the hypercohomology spectral sequence degenerates by coherent acyclicity:

$$H^p(W, \mathcal{H}^q(K)) \implies H^{p+q} \mathbf{R}\Gamma(W, K).$$

Hence

$$H^q \mathbf{R}\Gamma(W, K) = \Gamma(W, \mathcal{H}^q(K)).$$

The last module is finite. On degree- q cohomology, the counit is the finite-module isomorphism

$$\Gamma(W, \widetilde{\mathcal{H}^q(K)}) \xrightarrow{\sim} \mathcal{H}^q(K).$$

Thus the counit is an equivalence. The unit is an equivalence by the same calculation, proving $\mathrm{D}_{\mathrm{coh}}^b(W) \simeq \mathrm{D}_{\mathrm{fg}}^b(A)$.

Condensification identifies $\mathrm{D}_{\mathrm{pc}}(A)$ with the nuclear and pseudocompact subcategory [And21, Theorem 5.9, Proposition 5.22, and Theorem 5.50] [LH26, A.15]. Its heart is the image of finite modules and its bounded part is the image of $\mathrm{D}_{\mathrm{fg}}^b(A)$. This proves both assertions on W .

These equivalences commute with rational restriction [And21, Lemma 5.10]. Flatness makes the local truncation triangles compatible as well, so they descend to the chartwise t -structure. Apply Lemma 6.5 on the ordinary side and Proposition B.1, together with [And21, Theorems 5.3 and 5.44], on the solid side. A finite cover gives a common interval for bounded objects. Taking the limits gives the required equivalences, with exactness and the tensor structure on hearts inherited from the affinoid calculations. $\square$

Proposition 6.7. *Let $V = \mathrm{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid, and let T be a proper finitely presented R -scheme. Through Proposition 4.10, regard $\mathrm{D}_{\mathrm{pc}}^{\square}(T^{\mathrm{An}})$ as the chartwise pseudocoherent subcategory on $(T^{\mathrm{ad}/V})^{\square}$. The t -structure transported from $\mathrm{D}_{\mathrm{pc}}(T)$ by Proposition 4.14 agrees with the chartwise t -structure. Consequently there are canonical exact equivalences, the first symmetric monoidal,*

$$\begin{aligned} \beta_{T,V}^{\heartsuit} : \mathrm{D}_{\mathrm{pc}}^{\square}(T^{\mathrm{An}})^{\heartsuit} &\xrightarrow{\sim} \mathrm{Coh}(T^{\mathrm{ad}/V}), \\ \beta_{T,V}^b : \mathrm{D}_{\mathrm{pc}}^{\square}(T^{\mathrm{An}})^b &\xrightarrow{\sim} \mathrm{D}_{\mathrm{coh}}^b(T^{\mathrm{ad}/V}). \end{aligned}$$

For $\mathcal{F} \in \text{Coh}(T)$ and $M \in \text{D}_{\text{coh}}^b(T)$,

$$\begin{aligned}\beta_{T,V}^{\heartsuit}(\alpha_T^{\text{An}} \mathcal{F}) &\simeq c_{T/V}^* \mathcal{F}, \\ \beta_{T,V}^b(\alpha_T^{\text{An}} M) &\simeq \mathbf{L}c_{T/V}^* M.\end{aligned}$$

In particular, ordinary analytification induces equivalences

$$\begin{aligned}\text{Coh}(T) &\xrightarrow{\sim} \text{Coh}(T^{\text{ad}/V}), \\ \text{D}_{\text{coh}}^b(T) &\xrightarrow{\sim} \text{D}_{\text{coh}}^b(T^{\text{ad}/V}).\end{aligned}$$

Proof. Choose the fpqc family of Lemma 4.13, with $W_i = \text{Spa}(B_i, B_i^+)$ lying above $U_i = \text{Spec } A_i$. Proposition 4.10 gives

$$\alpha_T^{\text{An}}(M)|_{W_i^{\square}} \simeq \text{Cond}_{B_i}(B_i \otimes_{A_i}^{\mathbf{L}} \mathbf{R}\Gamma(U_i, M)).$$

Flatness of $A_i \rightarrow B_i$ implies

$$H^j(B_i \otimes_{A_i}^{\mathbf{L}} \mathbf{R}\Gamma(U_i, M)) \simeq B_i \otimes_{A_i} \Gamma(U_i, \mathcal{H}^j(M)).$$

The family is faithfully flat, so these modules detect the vanishing of $\mathcal{H}^j(M)$. Thus the transported and chartwise degree conditions agree. Finiteness of the family gives a common interval for bounded objects.

Proposition 6.6 now supplies $\beta_{T,V}^{\heartsuit}$ and $\beta_{T,V}^b$. The displayed scalar-extension formula identifies their composites with ordinary pullback $c_{T/V}^*$ and derived pullback $\mathbf{L}c_{T/V}^*$. Exactness and the tensor comparison on hearts hold on each affinoid and hence globally. Composing with Proposition 4.14 gives the final two equivalences. $\square$

Lemma 6.8. *Let $u : V' = \text{Spa}(R', R'^+) \rightarrow V = \text{Spa}(R, R^+)$ be a rational localization. The functors $\beta_{T,V}^{\heartsuit}$ and $\beta_{T,V}^b$ of Proposition 6.7 commute naturally with restriction to V' . For composable rational localizations, these equivalences satisfy the identity, composition, and higher associativity coherences.*

Proof. On an affinoid chart, each β is condensification followed by ordinary sheafification. Proposition 4.10 and scalar-extension compatibility identify both routes through $V' \subset V$ with the same iterated tensor product. Its associativity maps give the natural equivalence and its higher composition homotopies. By the formulas in Proposition 6.7, these are the usual restriction maps for c^* and $\mathbf{L}c^*$. $\square$

Proposition 6.9. *For proper $T \rightarrow X$, the local comparisons glue to a canonical exact symmetric monoidal equivalence on hearts and a canonical exact equivalence on bounded coherent complexes*

$$\begin{aligned}\beta_T^{\heartsuit} : \mathcal{P}_T^{\heartsuit} &\xrightarrow{\sim} \text{Coh}(T_{\mathcal{X}}^{\text{ad}}), \\ \beta_T^b : \mathcal{P}_T^b &\xrightarrow{\sim} \text{D}_{\text{coh}}^b(T_{\mathcal{X}}^{\text{ad}}).\end{aligned}$$

For $\mathcal{F} \in \text{Coh}(T)$ and $M \in \text{D}_{\text{coh}}^b(T)$ they satisfy

$$\begin{aligned}\beta_T^{\heartsuit}(\alpha_T \mathcal{F}) &\simeq c_T^* \mathcal{F}, \\ \beta_T^b(\alpha_T M) &\simeq \mathbf{L}c_T^* M.\end{aligned}$$

Proof. Lemma 6.8 makes the local $\beta_{T,V}$ compatible with rational restriction. We identify the hearts and bounded parts in the resulting descent diagrams.

Take $C \in \mathcal{P}_T$ with algebraic preimage $M \in D_{\text{pc}}(T)$. The condition that C lies in the transported heart is $\mathcal{H}^j(M) = 0$ for $j \neq 0$. It can be tested after the faithfully flat base change

$$T \times_X \coprod_i \text{Spec } R_i \longrightarrow T$$

of Lemma 2.7. There it says exactly that the restrictions lie in the local transported hearts. Thus $\mathcal{P}_T^\heartsuit$ is their limit. Coherent sheaves satisfy rational descent, so taking the limit of $\beta_{T,V}^\heartsuit$ gives $\beta_T^\heartsuit$.

For bounded data, choose an interval $[a, b]$ containing the bounds on this finite family. The rational-refinement argument of Lemma 6.5 gives the same interval on every chart. The algebraic preimage has cohomology only in $[a, b]$ after the fpqc pullback, and hence before it. Therefore $\mathcal{P}_T^b$ is the limit of the local bounded subcategories. On the ordinary side,

$$D_{\text{coh}}^b(T_{\mathcal{X}}^{\text{ad}}) \simeq \text{Cart}_{V \in \text{RatAff}_X(\mathcal{X})} D_{\text{coh}}^b(T_R^{\text{ad}/V})$$

by Lemma 6.5. Taking the limit of $\beta_{T,V}^b$ proves the bounded equivalence.

On affinoids the composites with α_T are the formulas for c_T^* and $\mathbf{L}c_T^*$. They agree under restriction and therefore give the stated identifications. The local t -structures give exactness, and the ordinary tensor products on coherent hearts give the symmetric monoidal structure. $\square$

Corollary 6.10. *For proper $T \rightarrow X$, ordinary analytification induces an exact symmetric monoidal equivalence*

$$c_T^* : \text{Coh}(T) \xrightarrow{\sim} \text{Coh}(T_{\mathcal{X}}^{\text{ad}})$$

and an exact equivalence of stable categories

$$\mathbf{L}c_T^* : D_{\text{coh}}^b(T) \xrightarrow{\sim} D_{\text{coh}}^b(T_{\mathcal{X}}^{\text{ad}}).$$

It also induces equivalences on perfect complexes and vector bundles.

Proof. Compose the equivalence α_T with Proposition 6.9. Its formulas identify the result with c_T^* and $\mathbf{L}c_T^*$, proving the first two assertions. Vector bundles are the dualizable objects in the coherent tensor category, so their comparison follows as well.

Derived pullback preserves perfect complexes. For the converse, first work over a base affinoid and take $M \in D_{\text{coh}}^b(T_R)$ whose analytification is perfect. A finite affinoid cover of the proper space $T_R^{\text{ad}/V}$ gives a common Tor-amplitude $[a, b]$. For every $N \in \text{Coh}(T_R)$, flatness gives

$$c_{T_R/V}^* H^j(M \otimes^{\mathbf{L}} N) \simeq H^j(\mathbf{L}c_{T_R/V}^* M \otimes^{\mathbf{L}} c_{T_R/V}^* N) = 0$$

for $j \notin [a, b]$. Each algebraic cohomology sheaf here is coherent, as one sees from a bounded-above finite-free resolution. Coherent GAGA therefore gives $H^j(M \otimes^{\mathbf{L}} N) = 0$ in the same degrees.

Let $U = \text{Spec } A \subset T_R$ and let Q be a finite A -module. Extend $\tilde{Q}$ to a coherent sheaf N on T_R [Stacks, Tag 0G41]. Restricting the preceding vanishing gives

$$H^j(\mathbf{R}\Gamma(U, M|_U) \otimes_A^{\mathbf{L}} Q) = 0$$

for $j \notin [a, b]$. Every A -module is the filtered union of its finite submodules. Derived tensor product preserves these colimits, and filtered colimits are exact. Hence the same vanishing

holds for every A -module. This gives finite Tor-amplitude. Since M is pseudocoherent, it is perfect [Stacks, Tag 0658].

For the global assertion, pull back along the finite fpqc family on X . If the analytification is perfect, the preceding argument makes every algebraic pullback perfect. Perfectness descends fpqc, so the original complex is perfect. The bounded equivalence therefore restricts to an equivalence on perfect complexes. $\square$

6.3. Ordinary proper pushforward. The comparison for direct images is local on the adic base. We use ordinary open base change to reduce it to the affinoid GAGA equivalence.

Lemma 6.11. *Let*

$$\begin{array}{ccc} Z' & \xrightarrow{j'} & Z \\ h' \downarrow & & \downarrow h \\ V' & \xrightarrow{j} & V \end{array}$$

be a cartesian square of ringed spaces in which j is an open immersion. Then the canonical morphism $j^* \mathbf{R}h_* \rightarrow \mathbf{R}h'_* j'^*$ is an equivalence on derived categories of modules.

Proof. Extension by zero along an open immersion is exact and left adjoint to restriction. The cartesian square gives $\mathbf{L}h^* j_! \simeq j'_! \mathbf{L}(h')^*$, as can be checked on opens. Thus

$$\begin{aligned} \mathrm{Map}_{V'}(B, j^* \mathbf{R}h_* A) &\simeq \mathrm{Map}_V(j_! B, \mathbf{R}h_* A) \\ &\simeq \mathrm{Map}_Z(\mathbf{L}h^* j_! B, A) \\ &\simeq \mathrm{Map}_Z(j'_! \mathbf{L}(h')^* B, A) \\ &\simeq \mathrm{Map}_{Z'}(\mathbf{L}(h')^* B, j'^* A) \\ &\simeq \mathrm{Map}_{V'}(B, \mathbf{R}h'_* j'^* A). \end{aligned}$$

These adjunctions identify the map with the canonical base-change mate. Yoneda proves the lemma. $\square$

Proposition 6.12. *Let $V = \mathrm{Spa}(R, R^+)$ be a strongly noetherian complete Tate affinoid, let $h : Z \rightarrow \mathrm{Spec} R$ be proper and of finite presentation, and let $h^{\mathrm{ad}} : Z^{\mathrm{ad}/V} \rightarrow V$ be its relative Huber analytification. For every $M \in \mathrm{D}_{\mathrm{coh}}^b(Z)$, the canonical comparison morphism*

$$\mathbf{L}c_V^* \mathbf{R}h_* M \rightarrow \mathbf{R}h_*^{\mathrm{ad}} \mathbf{L}c_{Z/V}^* M$$

is an equivalence. In particular, its target belongs to $\mathrm{D}_{\mathrm{coh}}^b(V)$. The construction is functorial in M and compatible with rational localization of V , including the higher composition coherences.

Proof. Analytify the algebraic counit to obtain

$$\mathbf{L}(h^{\mathrm{ad}})^* \mathbf{L}c_V^* \mathbf{R}h_* M \simeq \mathbf{L}c_{Z/V}^* \mathbf{L}h^* \mathbf{R}h_* M \rightarrow \mathbf{L}c_{Z/V}^* M.$$

Its adjoint is the comparison map.

For a rational $V' = \mathrm{Spa}(R', R'^+) \subset V$, put $Z_{R'} = Z \times_R \mathrm{Spec} R'$ and $M_{R'} = M \otimes_R^{\mathbf{L}} R'$. Flat base change, affinoid acyclicity, and Lemma 6.11 identify the map on sections over V' with

$$\mathbf{R}\Gamma(Z_{R'}, M_{R'}) \rightarrow \mathbf{R}\Gamma(Z_{R'}^{\mathrm{ad}/V'}, \mathbf{L}c_{Z/V}^* M_{R'}).$$

Bounded coherent GAGA over R' (Proposition 6.7) identifies the two sides as

$$\mathrm{Map}_{Z_{R'}}(\mathcal{O}, M_{R'}) \xrightarrow{\mathbf{L}c^*} \mathrm{Map}_{Z_{R'}^{\mathrm{ad}}/V'}(\mathcal{O}, \mathbf{L}c^* M_{R'}).$$

This is the canonical map just constructed: under adjunction, both maps come from analytifying

$$\mathbf{L}h_{R'}^* \mathbf{R}(h_{R'})_* M_{R'} \longrightarrow M_{R'}.$$

It is therefore an equivalence on sections over every rational V' .

Let Q be its cofiber and choose a with $M \in D^{\geq a}(Z)$. Flat analytification and left t -exactness of direct image give $Q \in D^{\geq a-1}(V)$. Lemma 6.2 identifies it with a left-bounded sheaf of module spectra. Its values on the rational basis are $\mathbf{R}\Gamma(V', Q|_{V'}) = 0$, so $Q = 0$. The source of the comparison is bounded coherent by proper coherence and flat pullback. Hence its target is bounded coherent as well.

Successive restrictions pull back the same counit. Naturality of mates and the associativity maps for flat and open base change give the stated compatibilities. $\square$

Proposition 6.13. *Let $f : Y \rightarrow X$ be proper and of finite presentation. For $T = X$, Proposition 3.5 identifies $X_{\mathcal{X}}^{\mathrm{ad}}$ with $\mathcal{X}$ and c_X with λ . For every $M \in D_{\mathrm{coh}}^b(Y)$, the canonical morphism*

$$\mathbf{L}c_X^* \mathbf{R}f_* M \longrightarrow \mathbf{R}f_*^{\mathrm{ad}} \mathbf{L}c_Y^* M$$

is an equivalence in $D_{\mathrm{coh}}^b(\mathcal{X})$.

Proof. Define the comparison as the adjoint of

$$\mathbf{L}(f^{\mathrm{ad}})^* \mathbf{L}c_X^* \mathbf{R}f_* M \simeq \mathbf{L}c_Y^* \mathbf{L}f^* \mathbf{R}f_* M \longrightarrow \mathbf{L}c_Y^* M.$$

For $V = \mathrm{Spa}(R, R^+) \subset \lambda^{-1}(W)$, with $W = \mathrm{Spec} A$, set $Y_R = Y \times_X \mathrm{Spec} R$ and $M_R = \mathbf{L}(Y_R \rightarrow Y)^* M$. The map $A \rightarrow R$ is flat. Flat base change, Proposition 6.12, and open base change give

$$\begin{aligned} (\mathbf{L}c_X^* \mathbf{R}f_* M)|_V &\simeq \mathbf{L}c_V^* \mathbf{R}(f_R)_* M_R \\ &\simeq \mathbf{R}(f_R^{\mathrm{ad}})_* \mathbf{L}c_{Y_R/V}^* M_R \\ &\simeq (\mathbf{R}f_*^{\mathrm{ad}} \mathbf{L}c_Y^* M)|_V. \end{aligned}$$

Each map comes from the restricted counit. Thus this is the restriction of the canonical global comparison.

On a finite rational-affinoid cover of $\mathcal{X}$, these comparisons are equivalences and their targets are bounded coherent. Taking the minimum lower bound and maximum upper bound gives a common interval. Coherence and cohomological vanishing are local, so the global target lies in $D_{\mathrm{coh}}^b(\mathcal{X})$. The cofiber of the global comparison vanishes on the cover, hence vanishes. This proves the proposition. $\square$

6.4. Classical GAGA.

Theorem 6.14. *Let $f : Y \rightarrow X$ be proper, and let $c_Y : Y_{\mathcal{X}}^{\mathrm{ad}} \rightarrow Y$ be its ordinary relative Huber analytification. The stable equivalence of Theorem 5.5 equips $D_{\mathrm{pc}}^{\square}(Y_{\mathcal{X}}^{\mathrm{An}})$ with the coherent t -structure transported from $D_{\mathrm{pc}}(Y)$. There is a canonical exact symmetric monoidal equivalence*

$$D_{\mathrm{pc}}^{\square}(Y_{\mathcal{X}}^{\mathrm{An}})^{\heartsuit} \xrightarrow{\sim} \mathrm{Coh}(Y_{\mathcal{X}}^{\mathrm{ad}}),$$

and a canonical exact equivalence of stable categories

$$D_{\text{pc}}^{\square}(Y_{\mathcal{X}}^{\text{An}})^b \xrightarrow{\sim} D_{\text{coh}}^b(Y_{\mathcal{X}}^{\text{ad}}),$$

under which the restriction of α_Y to coherent sheaves and bounded coherent complexes identifies with c_Y^* and $\mathbf{L}c_Y^*$, respectively. Consequently analytification induces equivalences

$$\begin{aligned} \text{Coh}(Y) &\xrightarrow{\sim} \text{Coh}(Y_{\mathcal{X}}^{\text{ad}}), \\ D_{\text{coh}}^b(Y) &\xrightarrow{\sim} D_{\text{coh}}^b(Y_{\mathcal{X}}^{\text{ad}}), \\ \text{Perf}(Y) &\xrightarrow{\sim} \text{Perf}(Y_{\mathcal{X}}^{\text{ad}}), \\ \text{Vect}(Y) &\xrightarrow{\sim} \text{Vect}(Y_{\mathcal{X}}^{\text{ad}}). \end{aligned}$$

For every $\mathcal{F} \in \text{Coh}(Y)$ and every $i \geq 0$, the canonical ordinary adic comparison maps are isomorphisms

$$\begin{aligned} (R^i f_* \mathcal{F})^{\text{ad}} &\xrightarrow{\sim} R^i f_*^{\text{ad}}(\mathcal{F}^{\text{ad}}), \\ H^i(Y, \mathcal{F}) &\xrightarrow{\sim} H^i(Y_{\mathcal{X}}^{\text{ad}}, \mathcal{F}^{\text{ad}}). \end{aligned}$$

Proof. Proposition 6.9 identifies the coherent heart and bounded part, and Corollary 6.10 gives the four equivalences. Apply Proposition 6.13 to $\mathcal{F}[0]$ and take cohomology sheaves to obtain the direct-image comparison.

By Lemma 5.4 and its cohomological-dimension bound, $\mathbf{R}f_* \mathcal{F}$ is bounded coherent. The base comparison [KL19, Theorem 4.7.5] therefore gives

$$\begin{aligned} \mathbf{R}\Gamma(Y, \mathcal{F}) &\simeq \mathbf{R}\Gamma(X, \mathbf{R}f_* \mathcal{F}) \\ &\simeq \mathbf{R}\Gamma_{\text{ad}}(\mathcal{X}, (\mathbf{R}f_* \mathcal{F})^{\text{ad}}) \\ &\simeq \mathbf{R}\Gamma_{\text{ad}}(\mathcal{X}, \mathbf{R}f_*^{\text{ad}} \mathcal{F}^{\text{ad}}) \\ &\simeq \mathbf{R}\Gamma_{\text{ad}}(Y_{\mathcal{X}}^{\text{ad}}, \mathcal{F}^{\text{ad}}). \end{aligned}$$

Taking cohomology proves the last assertion. $\square$

APPENDIX A. CONDENSED MATHEMATICS AND ANALYTIC STACKS

We recall the coefficient categories used in the paper, following [CS26, Sch26a, And21, LH26]. All complexes are cohomologically indexed; in particular, a connective complex has cohomology in degrees ≤ 0 . For related solid coefficient theories on relative Fargues–Fontaine curves, see [ALM26, AB+26].

A.1. Condensed sets and modules. A condensed set is a sheaf on profinite sets for finite jointly surjective families of continuous maps. The same definition gives condensed abelian groups, rings, and modules. For a Hausdorff topological space T , define

$$\underline{T}(S) = \text{Cont}(S, T).$$

This construction is fully faithful on compactly generated Hausdorff spaces [Sch26a, Lecture I]. For example, if B is a topological ring, then $\underline{B}(S)$ consists of continuous maps, while $B_{\text{disc}}(S)$ consists of locally constant maps with finite image.

A profinite set S is *extremally disconnected* if the closure of every open subset is open. Every profinite set is a continuous quotient of one of these. For a condensed ring A , the free modules $A[S]$ on extremally disconnected sets satisfy

$$\text{Hom}_A(A[S], M) = M(S).$$

They form compact projective generators of the abelian module category [And21, Section 2].

For an ordinary ring B , passage to discrete condensed modules gives a fully faithful functor

$$D(B) \longrightarrow D_{\text{cond}}(B) = D(\text{Mod}_{B_{\text{disc}}}^{\text{cond}}).$$

We use the category on the right when comparing analytic completeness conditions.

A.2. Derived operations and solid modules. Derived categories are regarded as stable ∞ -categories. Their mapping spectra retain the homotopies used in gluing diagrams and module structures. Finite limits and finite colimits agree, and a presentable stable category admits all small colimits. Tensor products are derived unless stated otherwise [Lur17, Sections 1.1 and 4.2].

An object P is *compact* if $\text{Map}(P, -)$ preserves filtered colimits. It is *dualizable* if it has evaluation and coevaluation maps satisfying the duality identities. In that case, with $P^\vee = \mathbf{R}\underline{\text{Hom}}(P, 1)$, we have

$$\mathbf{R}\underline{\text{Hom}}(P, N) \simeq P^\vee \otimes N.$$

In the derived category of an ordinary ring, the compact objects and the dualizable objects are both the perfect complexes. For the analytic categories below, we check compactness using their measure generators and dualizability using the tensor product.

Write a profinite set as $S = \lim_i S_i$, with each S_i finite, and put

$$\mathbf{Z}^\square[S] = \lim_i \mathbf{Z}[S_i].$$

A condensed abelian group M is *solid* if restriction gives a bijection

$$\text{Hom}(\mathbf{Z}^\square[S], M) \xrightarrow{\sim} \text{Hom}(\mathbf{Z}[S], M)$$

for every profinite S . The definition is independent of the chosen presentation. The inclusion of solid abelian groups has a left adjoint, called solidification. The derived solid tensor product is obtained by solidifying the condensed derived tensor product [Sch26a, Lecture II].

A.3. Analytic rings and complete modules. An analytic ring $\mathcal{A} = (A, \mathcal{M})$ consists of an animated condensed commutative ring A , connective measure modules $\mathcal{M}[S]$, and Dirac maps

$$A[S] \longrightarrow \mathcal{M}[S].$$

Here S runs over extremally disconnected profinite sets, and the measure functor takes finite disjoint unions to finite products. Animation allows the derived algebraic operations used in base change, including higher Tor groups [Sch26a, Sections 11–12].

An A -complex N is *$\mathcal{A}$ -complete* if

$$\mathbf{R}\text{Hom}_A(\mathcal{M}[S], N) \xrightarrow{\sim} \mathbf{R}\text{Hom}_A(A[S], N) = N(S)$$

for every such S . The analyticity axiom requires completeness for the connective objects generated by the measure modules under sifted colimits. In the nonanimated case, it can be tested on bounded-above complexes of direct sums of measure modules [And21, Definitions 2.5 and 2.14]. We also use the Frobenius-compatibility condition for animated commutative rings from [Sch26a, Definition 12.10].

The complete objects form a full presentable stable subcategory $D(\mathcal{A})$ of condensed A -complexes. Its inclusion has a left adjoint, analytic completion, and its tensor product is the completed derived tensor product. The measure modules are compact projective generators

in the connective category and compact generators after stabilization. For a complete analytic ring, $A \simeq \mathcal{M}[*]$, so the unit is compact [Sch26a, Proposition 12.4, Remark 12.5, and Definition 12.9].

For a map $\mathcal{A} \rightarrow \mathcal{B}$, pullback is scalar extension followed by analytic completion. It is exact, strong symmetric monoidal, and preserves colimits. Restriction of scalars is its right adjoint.

Suppose that a connective animated A -algebra B is already $\mathcal{A}$ -complete. Its induced analytic structure satisfies

$$D(\mathcal{B}) \simeq \text{Mod}_B(D(\mathcal{A})).$$

Thus a B -module is complete exactly when its underlying $\mathcal{A}$ -module is complete. The underlying ring and the reflector onto complete connective modules determine a complete analytic ring [Sch26a, Propositions 12.8 and 12.20]. We use this description to identify analytic-ring pushouts in Appendix B.

A.4. Huber pairs and relative condensification. A Huber ring has an open ring of definition with a finitely generated ideal of definition. It is Tate if it contains a topologically nilpotent unit. A Huber pair (R, R^+) specifies an open integrally closed subring of power-bounded elements. Strong noetherianity means that every Tate algebra $R\langle T_1, \dots, T_n \rangle$ is noetherian.

A complete Tate Huber pair gives an analytic ring $(R, R^+)^\square$ over $\mathbf{Z}^\square$ [And21, Theorem 3.28]. We write

$$D_\square(R, R^+) = D((R, R^+)^\square).$$

For $M \in D(R)$, define relative condensification by

$$\text{Cond}_R(M) = \underline{R} \otimes_{R_{\text{disc}}}^{\mathbf{L}} M_{\text{disc}}.$$

This condensed tensor product is already complete. The functor is fully faithful, exact, strong symmetric monoidal, and preserves colimits [And21, Theorem 5.9 and Lemma 5.10]. Objects in its essential image are called *relatively discrete*. For example, the topological coefficient ring itself is $\underline{R} = \text{Cond}_R(R)$.

For a rational localization $(R, R^+) \rightarrow (R', R'^+)$ between strongly noetherian complete Tate pairs,

$$\text{Cond}_R(M)|_{\text{Spa}(R', R'^+)} \simeq \text{Cond}_{R'}(R' \otimes_R^{\mathbf{L}} M).$$

Since $R \rightarrow R'$ is flat, cohomology and truncations commute with this restriction. Lemmas B.7 and B.8 describe how the completeness condition changes with the plus ring.

A.5. Analytic affines, steady maps, and descent. The analytic affine associated to $\mathcal{A}$ is denoted $\text{AnSpec } \mathcal{A}$. The construction is contravariant. A morphism $\mathcal{A} \rightarrow \mathcal{B}$ is a localization if restriction of scalars on complete modules is fully faithful. Steadiness is the base-change condition on analytic completion: for a cartesian square of analytic affines

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{g'} & \mathcal{U} \\ f' \downarrow & & \downarrow f \\ \mathcal{V} & \xrightarrow{g} & \mathcal{Z}, \end{array}$$

a steady ring map defining f gives

$$f^* \mathbf{R}g_* \xrightarrow{\sim} \mathbf{R}g'_* f'^*.$$

Steady maps are stable under composition and base change [Sch26a, Definitions 12.13 and 12.16 and Propositions 12.14–12.15].

An analytic stack is a sheaf of spaces on analytic affines for finite jointly conservative families of steady localizations. Rational affinoid covers used in this paper give such families. Complete-module categories glue by

$$\mathcal{C}_\square(\mathcal{Z}) = \lim_{(\mathrm{AnSpec} \mathcal{A} \rightarrow \mathcal{Z})^{\mathrm{op}}} \mathbf{D}(\mathcal{A}).$$

In particular, for an effective epimorphism $\mathcal{U}_0 \rightarrow \mathcal{Z}$ and its Čech nerve $\mathcal{U}_\bullet$,

$$\mathcal{C}_\square(\mathcal{Z}) \simeq \lim_{[n] \in \Delta} \mathcal{C}_\square(\mathcal{U}_n).$$

Concretely, an object is a complex on each chart with pullback equivalences and compatible homotopies on the intersections. Proposition B.1 proves this statement and its version for modules. For a finite open cover, repeated two-open Mayer–Vietoris gives the finite calculation used in Proposition B.6.

A.6. The algebraic, analytic, and adic realizations. Let (R, R^+) be a complete Tate pair, and let B be a finitely presented R -algebra. Write B^+ for the integral closure of the image of R^+ in B . The algebraic realization is defined by

$$B_R^{\mathrm{alg}} = (R, R^+)^\square \otimes_{(R_{\mathrm{disc}}, R_{\mathrm{disc}}^+)^\square}^{\mathrm{AnRing}} (B_{\mathrm{disc}}, B_{\mathrm{disc}}^+)^\square.$$

By Lemma B.10,

$$\mathcal{C}_\square((\mathrm{Spec} B)^{\mathrm{Alg}}) \simeq \mathrm{Mod}_{\mathrm{Cond}_R(B)}(\mathbf{D}_\square(R, R^+)).$$

An object on this realization is therefore a complete coefficient object with a B -action. Direct image to the base forgets the action.

Choose generators $b_1, \dots, b_m$ and a topologically nilpotent unit ϖ . The analytic locus $(\mathrm{Spec} B)^{\mathrm{An}}$ is the union of the bounded charts defined by $|\varpi^n b_i| \leq 1$ for $1 \leq i \leq m$. For $B = R[T]$ over a strongly noetherian base, these are the discs of radii $|\varpi|^{-n}$. Lemma B.9 compares their choices and their rational base changes. Both realizations extend to separated finitely presented schemes by Zariski gluing [LH26, A.9–A.13].

Huber analytification T^{ad} is an adic space. Gluing

$$\mathrm{Spa}(C, C^+) \mapsto \mathrm{AnSpec}(C, C^+)^\square$$

gives its associated analytic stack $(T^{\mathrm{ad}})^\square$. Over a strongly noetherian base, Proposition 4.10 identifies

$$T^{\mathrm{An}} \simeq (T^{\mathrm{ad}})^\square$$

and compares their coefficient categories. For proper T , the valuative criterion also identifies T^{An} with T^{Alg} [LH26, Theorem A.13]. This allows the boundary argument to combine proper analytic direct image with localization on the algebraic realization.

A.7. Perfectness, pseudocoherence, and nuclearity. An ordinary complex is perfect if it has a bounded resolution by finite projective modules. It is pseudocoherent if it has

a bounded-above resolution of this kind. On schemes these conditions are local. For a noetherian scheme T ,

$$D_{\text{pc}}(T) = D_{\text{coh}}^-(T),$$

and a pseudocoherent complex with finite Tor-amplitude is perfect [Stacks, Tags 064Q and 0656].

For example, over $A = k[\epsilon]/(\epsilon^2)$ the module k has the resolution

$$\cdots \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \longrightarrow k \longrightarrow 0.$$

It is pseudocoherent and has infinite projective dimension.

In the nonanimated Huber coefficient categories used here, pseudocompact objects have bounded-above resolutions by compact projectives. The equivalent mapping criterion is that, for every uniformly bounded-below family (N_λ) ,

$$\bigoplus_{\lambda} \mathbf{R} \operatorname{Hom}(P, N_\lambda) \xrightarrow{\sim} \mathbf{R} \operatorname{Hom}\left(P, \bigoplus_{\lambda} N_\lambda\right)$$

[CS26, Definition 9.9].

For $\mathcal{A} = (A, \mathcal{M})$, nuclearity is the condition that

$$(\mathcal{M}[S]^\vee \otimes_{\mathcal{A}}^{\mathbf{L}} N)(*) \xrightarrow{\sim} N(S)$$

for every extremally disconnected S , using derived duals and evaluation [And21, Definition 5.30]. Nuclear objects are closed under colimits. On the complete Tate affinoids in this paper, the two finiteness conditions together give

$$\operatorname{Cond}_R(D_{\text{pc}}(R)) = \{N \in D_{\square}(R, R^+) : N \text{ is nuclear and pseudocompact}\}$$

[And21, Theorem 5.50] [LH26, A.15]. On relative analytic realizations, they are tested on the two systems of bounded charts in [LH26, A.15]. Their intersection is $D_{\text{pc}}^{\square}$. Proposition 4.10 describes this category through ordinary pseudocoherent complexes on Huber affinoids; flat rational restriction lets their coherent t -structures glue.

APPENDIX B. CATEGORICAL AND ANALYTIC PRELIMINARIES

We collect the descent and base-change arguments used in the proof. The notation is that of Appendix A; the analytic-ring constructions are from [Sch26a, Section 12] and [LH26, A.1–A.3].

B.1. Descent and fully faithful functors.

Proposition B.1. *Let $\mathcal{C} = \operatorname{AnRing}^{\text{op}}$ with the topology $\mathcal{D}$ generated by finite families of jointly conservative steady localizations, and let $\operatorname{Shv}_{\mathcal{D}}(\mathcal{C})$ be the resulting ∞ -topos of analytic stacks. Let $a : \mathcal{U}_0 \rightarrow \mathcal{Z}$ be an effective epimorphism in this topos, and let $\mathcal{U}_{\bullet}$ be its Čech nerve. Then restriction is an equivalence in symmetric monoidal presentable stable ∞ -categories*

$$\mathcal{C}_{\square}(\mathcal{Z}) \xrightarrow{\sim} \lim_{[n] \in \Delta} \mathcal{C}_{\square}(\mathcal{U}_n).$$

More generally, for every small diagram $(\mathcal{Z}_i)_{i \in I}$ of analytic stacks, there is a natural equivalence

$$\mathcal{C}_\square \left(\operatorname{colim}_{i \in I}^{\operatorname{Shv}_{\mathcal{D}}} \mathcal{Z}_i \right) \xrightarrow{\sim} \lim_{i \in I^{\operatorname{op}}} \mathcal{C}_\square(\mathcal{Z}_i).$$

In particular, for every simplicial analytic stack $\mathcal{V}_\bullet$, $\mathcal{C}_\square(|\mathcal{V}_\bullet|) \simeq \lim_{[n] \in \Delta} \mathcal{C}_\square(\mathcal{V}_n)$. Colimits in each category on the right are computed degreewise. For every $A \in \operatorname{CAlg}(\mathcal{C}_\square(\mathcal{Z}))$, restriction also gives an equivalence

$$\operatorname{Mod}_A(\mathcal{C}_\square(\mathcal{Z})) \xrightarrow{\sim} \lim_{[n] \in \Delta} \operatorname{Mod}_{A|_{\mathcal{U}_n}}(\mathcal{C}_\square(\mathcal{U}_n)).$$

This applies to the simplicial gluing diagrams defining algebraic realizations [LH26, Theorem A.13].

Proof. Let D_0 be the affine complete-module functor

$$\begin{aligned} \mathcal{C}^{\operatorname{op}} &\longrightarrow \operatorname{CAlg}(\operatorname{Pr}_{\operatorname{st}}^L), \\ \operatorname{AnSpec}(A, M) &\longmapsto D(A, M). \end{aligned}$$

It satisfies descent for the generating covers [LH26, A.2] [Sch26a, Proposition 12.18]. Extend it to presheaves by

$$\overline{D}(F) = \lim_{(U \rightarrow F) \in (\mathcal{C}/F)^{\operatorname{op}}} D_0(U).$$

This right Kan extension carries presheaf colimits to limits, since the presheaf category is the free cocompletion of $\mathcal{C}$ [Sch26b, Proposition 5.16].

Write L for sheafification. Affine descent implies that $\overline{D}$ inverts covering-sieve inclusions. These generate the sheafification equivalences as a strongly saturated class [Lur09, Section 6.2.2]. The maps inverted by $\overline{D}$ are closed under two-out-of-three and retracts, and under pushouts and transfinite compositions because these become pullbacks and inverse limits of equivalences. Thus $\overline{D}$ factors through L , giving the stack-level functor $\mathcal{C}_\square$. We obtain

$$\begin{aligned} \mathcal{C}_\square \left(\operatorname{colim}_i^{\operatorname{Shv}} \mathcal{Z}_i \right) &\simeq \overline{D} \left(\operatorname{colim}_i^{\operatorname{PSh}} \mathcal{Z}_i \right) \\ &\simeq \lim_{i \in I^{\operatorname{op}}} \overline{D}(\mathcal{Z}_i) \simeq \lim_{i \in I^{\operatorname{op}}} \mathcal{C}_\square(\mathcal{Z}_i). \end{aligned}$$

For completeness, consider a generating cover $(U_i \rightarrow U)$ and its generated sieve S . The map to the representable factors as

$$\coprod_i y(U_i) \longrightarrow S \hookrightarrow y(U).$$

The first map is an effective epimorphism: S is the (-1) -truncated image of the coproduct map [Lur09, Lemma 6.2.3.18]. Sheafification is left exact and preserves colimits, hence preserves the Čech nerve and its realization. Also $LS \simeq Ly(U)$ because S is covering. Therefore

$$\coprod_i Ly(U_i) \longrightarrow Ly(U)$$

is an effective epimorphism [Lur09, Lemma 6.2.3.19]. For a cover by open substacks, the image of the coproduct is their union. Its equality with the target gives the same conclusion. These statements persist after base change.

For the effective epimorphism a in the proposition, we have $|\mathcal{U}_\bullet| \simeq \mathcal{Z}$ [Lur09, Corollary 6.2.3.5]. Applying the colimit formula proves the Čech assertion. Applied to any simplicial diagram, the same formula proves the geometric-realization assertion.

Finally, pullbacks preserve colimits. Componentwise colimits of cartesian sections therefore retain their pullback equivalences, which computes colimits in the limit category. Commutative algebra objects and modules over a cartesian algebra are also formed componentwise [Lur17, Sections 3.2.2 and 4.2.3]. This gives the module equivalence. $\square$

Lemma B.2. *Let $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ be cosimplicial diagrams of ∞ -categories and let $F^\bullet : \mathcal{C}^\bullet \rightarrow \mathcal{D}^\bullet$ be a morphism of diagrams. It induces a functor on limits. If every F^n is fully faithful, respectively an equivalence, then the induced functor is fully faithful, respectively an equivalence. If every F^n is fully faithful, a cartesian section of $\mathcal{D}^\bullet$ lies in the essential image whenever each of its components lies in the essential image of F^n . Natural transformations descend to limits. If the diagrams are presentable, the transition functors preserve colimits, and colimits in the limits are computed degreewise, then degreewise colimit preservation descends as well.*

In addition, if $F : \mathcal{C} \rightarrow \mathcal{D}$ is fully faithful and I is any small ∞ -category, then $\mathrm{Fun}(I, \mathcal{C}) \rightarrow \mathrm{Fun}(I, \mathcal{D})$ is fully faithful, with essential image the diagrams whose values lie in the essential image of F .

Proof. For $x = (x_n)$ and $y = (y_n)$ in the source limit,

$$\mathrm{Map}_{\lim \mathcal{C}^\bullet}(x, y) \simeq \lim_{[n] \in \Delta} \mathrm{Map}_{\mathcal{C}^n}(x_n, y_n).$$

If every F^n is fully faithful, the maps between the terms are equivalences, so the induced map on limits is an equivalence. Degreewise equivalences likewise give an equivalence of the limit categories.

Let $\mathcal{E}^n$ be the full essential image of F^n . Naturality makes $(\mathcal{E}^n)$ a full cosimplicial subdiagram of $(\mathcal{D}^n)$, and

$$\lim \mathcal{C}^\bullet \simeq \lim \mathcal{E}^\bullet.$$

A cartesian section with values in $\mathcal{E}^n$ has all its maps and coherence data in that full subdiagram. It therefore lifts to the source. Natural transformations and, under the stated hypotheses, colimits are computed componentwise.

For the last assertion, write $\mathcal{E} \subset \mathcal{D}$ for the full essential image of F . Then

$$\mathrm{Fun}(I, \mathcal{C}) \simeq \mathrm{Fun}(I, \mathcal{E}) \subset \mathrm{Fun}(I, \mathcal{D}).$$

The last inclusion is full. A diagram factors through it precisely when its values lie in $\mathcal{E}$. $\square$

Lemma B.3. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful strong symmetric monoidal functor between presentable symmetric monoidal stable ∞ -categories, and let $A \in \mathrm{CAlg}(\mathcal{C})$. Then $\mathrm{Mod}_A(\mathcal{C}) \rightarrow \mathrm{Mod}_{F(A)}(\mathcal{D})$ is fully faithful. Its essential image consists exactly of the modules whose underlying object lies in the essential image of F .*

Proof. Suppose that an $F(A)$ -module N has underlying object $F(M)$. Its action is a map $F(A) \otimes F(M) \rightarrow F(M)$. Since F is fully faithful and strong symmetric monoidal,

$$\mathrm{Map}_{\mathcal{C}}(A \otimes M, M) \simeq \mathrm{Map}_{\mathcal{D}}(F(A) \otimes F(M), F(M)).$$

The action therefore lifts to $\mu : A \otimes M \rightarrow M$.

Associativity is a specified homotopy between $\mu \circ (m_A \otimes \text{id}_M)$ and $\mu \circ (\text{id}_A \otimes \mu)$. Both maps lie in $\text{Map}_{\mathcal{C}}(A \otimes A \otimes M, M)$, so full faithfulness lifts the homotopy. The unit homotopy lifts in the same way.

For every $r \geq 0$, we also have

$$\text{Map}_{\mathcal{C}}(A^{\otimes r} \otimes M, M) \simeq \text{Map}_{\mathcal{D}}(F(A)^{\otimes r} \otimes F(M), F(M)).$$

These equivalences respect composition and insertion of units. Thus they identify the operation spaces for the two-colored module operad $\text{LM}^{\otimes}$ with A fixed, and lift all the higher module relations [Lur17, Sections 2.2.1 and 4.2.1]. We have obtained an A -module whose image is N .

Replacing the target M in these operation spaces by M' gives the same comparison for module maps and their homotopies. This proves full faithfulness. Conversely, a module in the image has its underlying object in the image of F , proving the essential-image assertion. $\square$

Lemma B.4. *Let $\mathcal{C}^{\bullet}$ and $\mathcal{D}^{\bullet}$ be cosimplicial diagrams of presentable stable ∞ -categories, and let $F^{\bullet} : \mathcal{C}^{\bullet} \rightarrow \mathcal{D}^{\bullet}$ be a morphism of diagrams. Suppose that every F^n admits a right adjoint G^n and that every square determined by a morphism of Δ is right-adjointable. Then the right-adjoint mates equip the G^n with a coherent natural transformation $\mathcal{D}^{\bullet} \rightarrow \mathcal{C}^{\bullet}$, indexed by the same category Δ , and the induced functor $F : \lim_{[n] \in \Delta} \mathcal{C}^n \rightarrow \lim_{[n] \in \Delta} \mathcal{D}^n$ admits the degreewise right adjoint $G : \lim_{[n] \in \Delta} \mathcal{D}^n \rightarrow \lim_{[n] \in \Delta} \mathcal{C}^n$.*

Proof. Let $\theta : [m] \rightarrow [n]$, and write c_{θ} and d_{θ} for the transition functors. If η^n and ε^n are the unit and counit of $F^n \dashv G^n$, the mate is

$$\begin{aligned} \beta_{\theta} : c_{\theta} G^m &\xrightarrow{\eta^n} G^m F^n c_{\theta} G^m \\ &\simeq G^m d_{\theta} F^m G^m \xrightarrow{G^n d_{\theta} \varepsilon^m} G^n d_{\theta}. \end{aligned}$$

Right-adjointability says that β_{θ} is an equivalence.

For a cartesian section $y = (y_n)$, use the maps

$$c_{\theta} G^m y_m \xrightarrow{\beta_{\theta}} G^n d_{\theta} y_m \xrightarrow{\sim} G^n y_n.$$

For composable restrictions, the two formulas agree by the triangle identities: the intermediate unit and counit cancel. Naturality carries the higher homotopies as well. Thus $(G^n y_n)$ is a cartesian section, functorially in y .

The local adjunctions commute with these structure maps. Taking their limit gives

$$\begin{aligned} \text{Map}_{\lim \mathcal{D}^{\bullet}}(Fx, y) &\simeq \lim_{[n] \in \Delta} \text{Map}_{\mathcal{D}^n}(F^n x_n, y_n) \\ &\simeq \lim_{[n] \in \Delta} \text{Map}_{\mathcal{C}^n}(x_n, G^n y_n) \\ &\simeq \text{Map}_{\lim \mathcal{C}^{\bullet}}(x, Gy), \end{aligned}$$

which proves the adjunction. $\square$

B.2. Finite open-cover descent. For a two-open cover, descent gives a homotopy pullback of categories. We use the resulting formula for mapping spectra.

Lemma B.5. *Let $\mathcal{Z} = \mathcal{U} \cup \mathcal{V}$ be an open cover by analytic substacks for which solid quasicoherent complexes satisfy descent. For $K, L \in \mathcal{C}_{\square}(\mathcal{Z})$, there is a natural cartesian*

square of spectra

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Z}}(K, L) & \longrightarrow & \mathrm{Map}_{\mathcal{U}}(K|_{\mathcal{U}}, L|_{\mathcal{U}}) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{V}}(K|_{\mathcal{V}}, L|_{\mathcal{V}}) & \longrightarrow & \mathrm{Map}_{\mathcal{U} \cap \mathcal{V}}(K|_{\mathcal{U} \cap \mathcal{V}}, L|_{\mathcal{U} \cap \mathcal{V}}). \end{array}$$

Equivalently, the first term is the fiber of the difference of the two restriction maps from the direct sum of the middle terms to the last term.

Proof. The coproduct $\mathcal{U} \amalg \mathcal{V} \rightarrow \mathcal{Z}$ is an effective epimorphism by Proposition B.1, since its image is the union of the two opens. In the sheaf topos,

$$\mathcal{Z} \simeq \mathcal{U} \coprod_{\mathcal{U} \cap \mathcal{V}} \mathcal{V}.$$

To check this, pull the pushout back to each covering open. The result is that open itself, and the cover detects equivalences.

Proposition B.1 now gives

$$\mathcal{C}_{\square}(\mathcal{Z}) \simeq \mathcal{C}_{\square}(\mathcal{U}) \times_{\mathcal{C}_{\square}(\mathcal{U} \cap \mathcal{V})} \mathcal{C}_{\square}(\mathcal{V}).$$

A map from K to L is a pair of local maps together with a homotopy between their restrictions to the intersection. The mapping space is therefore the corresponding homotopy pullback. Applying this to shifts gives the asserted square of spectra. In spectra, the pullback of $A \rightarrow C \leftarrow B$ is the fiber of the difference map $A \oplus B \rightarrow C$, which proves the last assertion. $\square$

Proposition B.6. *Let $\mathcal{Z}$ be an analytic stack with a finite open cover $\mathcal{Z} = \bigcup_{i=1}^r \mathcal{U}_i$. For every nonempty subset $I \subseteq \{1, \dots, r\}$, put $\mathcal{U}_I = \bigcap_{i \in I} \mathcal{U}_i$. Assume that every $\mathcal{U}_I$ is empty or a finite disjoint union of analytic affines $\mathrm{AnSpec}(R, R^+)^\square$ attached to complete Huber pairs. Then:*

- (i) $\mathbf{R}\Gamma(\mathcal{Z}, -)$ commutes with filtered colimits;
- (ii) if $K \in \mathcal{C}_{\square}(\mathcal{Z})$ restricts to a compact object of $\mathcal{C}_{\square}(\mathcal{U}_I)$ for every nonempty I , then K is compact in $\mathcal{C}_{\square}(\mathcal{Z})$.

Proof. On $\mathrm{AnSpec}(R, R^+)^\square$, global sections are maps from the analytic unit. The unit is the measure module of the one-point profinite set and is compact [And21, Theorem 3.28 and Proposition 2.11]. Hence global sections preserve filtered colimits. The same holds on a finite disjoint union, where both the category and global sections are finite products. The empty intersection contributes zero.

Induct on r . The case $r = 1$ follows from this calculation. For $r > 1$, put

$$\begin{aligned} \mathcal{Z}' &= \bigcup_{i < r} \mathcal{U}_i, \\ \mathcal{W} &= \mathcal{Z}' \cap \mathcal{U}_r = \bigcup_{i < r} (\mathcal{U}_i \cap \mathcal{U}_r). \end{aligned}$$

The covers of $\mathcal{Z}'$ and $\mathcal{W}$ have at most $r - 1$ members. Their multiple intersections are respectively $\mathcal{U}_I$ and $\mathcal{U}_{I \cup \{r\}}$, so the induction hypotheses apply.

By Lemma B.5, global sections on $\mathcal{Z}$ are the fiber of

$$\mathbf{R}\Gamma(\mathcal{Z}', L|_{\mathcal{Z}'}) \oplus \mathbf{R}\Gamma(\mathcal{U}_r, L|_{\mathcal{U}_r}) \longrightarrow \mathbf{R}\Gamma(\mathcal{W}, L|_{\mathcal{W}}).$$

The three functors preserve filtered colimits by induction, as do the restrictions. Filtered colimits of spectra preserve fibers, proving (i).

For (ii), induction makes $K|_{\mathcal{Z}'}$ and $K|_{\mathcal{W}}$ compact, while $K|_{\mathcal{U}_r}$ is compact by assumption. Apply the same fiber calculation to $\text{Map}(K, -)$. Each local mapping functor preserves filtered colimits, so their fiber does as well. Thus K is compact. $\square$

B.3. Integral structures and bounded charts. Fix a complete Tate Huber pair (R, R^+) . For a finitely presented R -algebra B , let B^+ be the integral closure of the image of R^+ in B . By [LH26, A.9], its algebraic realization is defined by

$$B_R^{\text{alg}} = (R, R^+)^{\square} \otimes_{(R_{\text{disc}}, R_{\text{disc}}^+)^{\square}}^{\text{AnRing}} (B_{\text{disc}}, B_{\text{disc}}^+)^{\square}.$$

Thus $(\text{Spec } B)^{\text{Alg}} = \text{AnSpec } B_R^{\text{alg}}$. Its analytic locus is the union of the bounded charts associated to a pseudo-uniformizer and algebra generators. For separated finitely presented schemes, both constructions are defined by Zariski gluing; they agree for proper schemes [LH26, A.9–A.13].

Lemma B.7. *Let $R \rightarrow B$ be a ring map, let $R^+ \subset R$ be an integrally closed subring, and let B^+ be the integral closure of the image of R^+ in B . Put*

$$\begin{aligned} \mathcal{R}_0 &= (R_{\text{disc}}, R_{\text{disc}}^+)^{\square}, \\ \mathcal{B}_0 &= (B_{\text{disc}}, B_{\text{disc}}^+)^{\square}. \end{aligned}$$

The discrete condensed ring B is $\mathcal{R}_0$ -complete. The analytic ring induced from $\mathcal{R}_0$ along $R \rightarrow B$ is therefore already complete and is canonically equivalent to $\mathcal{B}_0$. Consequently restriction of scalars gives a canonical symmetric monoidal equivalence

$$\text{D}(\mathcal{B}_0) \xrightarrow{\sim} \text{Mod}_B(\text{D}(\mathcal{R}_0)).$$

Under the forgetful functors to $\text{D}_{\text{cond}}(B)$, both sides are the full subcategory of condensed B -complexes whose underlying condensed R -complex is $\mathcal{R}_0$ -complete. The construction is functorial in maps of such integral pairs.

Proof. Let $N \in \text{D}_{\text{cond}}(B)$, and factor the map on integral subrings as

$$R^+ \twoheadrightarrow C^+ := \text{im}(R^+ \rightarrow B) \longrightarrow B^+.$$

Both arrows are integral. The restriction criterion [And21, Proposition 3.21] and integral invariance [And21, Proposition 3.22] give

$$\begin{aligned} N \text{ is } \mathcal{B}_0\text{-complete} &\iff N|_{B^+} \text{ is } (B_{\text{disc}}^+)^{\square}\text{-complete} \\ &\iff N|_{C^+} \text{ is } (C_{\text{disc}}^+)^{\square}\text{-complete} \\ &\iff N|_{R^+} \text{ is } (R_{\text{disc}}^+)^{\square}\text{-complete} \\ &\iff N|_R \text{ is } \mathcal{R}_0\text{-complete.} \end{aligned}$$

In particular, apply this to $N = B[0]$. The discrete Huber pair (B, B^+) has complete unit [And21, Lemma 3.24]. Thus B is already $\mathcal{R}_0$ -complete.

The induced analytic structure $\mathcal{B}_{\text{ind}}$ on B therefore has complete unit and

$$\text{D}(\mathcal{B}_{\text{ind}}) \simeq \text{Mod}_B(\text{D}(\mathcal{R}_0))$$

by [Sch26a, Proposition 12.8] and [LH26, Lemma A.1 and A.3]. Its complete objects are precisely the condensed B -complexes whose underlying R -complex is $\mathcal{R}_0$ -complete. The

calculation above identifies these with the $\mathcal{B}_0$ -complete objects. Since both complete analytic rings have underlying ring B , [Sch26a, Proposition 12.20] gives

$$\mathcal{B}_{\text{ind}} \simeq \mathcal{B}_0.$$

This proves the module equivalence.

The analytic structure is recovered from the reflector onto its complete connective modules. Such a reflector is unique up to contractible choice. For a map of integral pairs, the two composites induce the same ring map and restriction functor, so this uniqueness identifies them. It also gives compatibility for longer chains and proves the stated functoriality. $\square$

Lemma B.8. *Let $(R, R^+) \rightarrow (R', R'^+)$ be a rational localization of strongly noetherian complete Tate Huber pairs, let B be a finitely presented R -algebra, and put $B' = B \otimes_R R'$. Let B^+ be the integral closure of the image of R^+ in B , and let B'^+ be the integral closure of the image of R'^+ in B' . Then there is a canonical equivalence of complete analytic rings with discrete underlying condensed rings*

$$(R'_{\text{disc}}, R'^+_{\text{disc}})^{\square} \otimes_{(R_{\text{disc}}, R^+_{\text{disc}})^{\square}}^{\text{AnRing}} (B_{\text{disc}}, B^+_{\text{disc}})^{\square} \simeq (B'_{\text{disc}}, B'^+_{\text{disc}})^{\square}.$$

For composable rational localizations, these equivalences commute with the associativity identifications for analytic-ring pushouts.

Proof. Write $\mathcal{R}_0$, $\mathcal{B}_0$, $\mathcal{R}'_0$, and $\mathcal{B}'_0$ for the four discrete integral-pair analytic rings, and put

$$\mathcal{P} = \mathcal{R}'_0 \otimes_{\mathcal{R}_0}^{\text{AnRing}} \mathcal{B}_0.$$

The image of B^+ in B' is integral over the image of R'^+ : its monic equations over R^+ remain monic after base change. Thus $B \rightarrow B'$ carries B^+ into B'^+ and defines a comparison $\mathcal{P} \rightarrow \mathcal{B}'_0$.

Before analytic completion, the derived underlying ring is

$$B_{\text{disc}} \otimes_{R_{\text{disc}}}^{\mathbf{L}} R'_{\text{disc}} \simeq B'_{\text{disc}}[0].$$

Here we use flatness of the rational localization [KL19, Lemma 2.2.6] and compatibility of discrete condensification with derived tensor products.

By [Sch26a, Proposition 12.12], a connective condensed B' -complex is $\mathcal{P}$ -complete when its restrictions to R' and B are $\mathcal{R}'_0$ -complete and $\mathcal{B}_0$ -complete. The first condition implies the second. Indeed, restriction makes an $\mathcal{R}'_0$ -complete object $\mathcal{R}_0$ -complete, and Lemma B.7 then makes its B -restriction $\mathcal{B}_0$ -complete. Applying the same lemma over R' gives

$$N \text{ is } \mathcal{P}\text{-complete} \iff N|_{R'} \text{ is } \mathcal{R}'_0\text{-complete} \iff N \text{ is } \mathcal{B}'_0\text{-complete}.$$

The unit $B'[0]$ is complete by that lemma as well. Analytic completion therefore preserves the underlying ring, and [Sch26a, Proposition 12.20] identifies the comparison with an equivalence.

For $R \rightarrow R' \rightarrow R''$, associativity identifies the direct and iterated pushout constructions. The preceding calculation gives the same underlying ring $B \otimes_R R''$ and the same completeness condition over $(R''_{\text{disc}}, R''^+_{\text{disc}})^{\square}$. Uniqueness of the reflector identifies the two equivalences and gives the compatibility for longer chains. $\square$

We next compare the bounded charts under rational base change. For $B = R[T]$, the chart at level n is the disc $|T| \leq |\varpi|^{-n}$.

Lemma B.9. *Let $(R, R^+) \rightarrow (R', R'^+)$ be a rational localization of strongly noetherian complete Tate Huber pairs. Let B be a finitely presented R -algebra, choose a topologically nilpotent unit $\varpi \in R$ and a finite set of R -algebra generators $\mathbf{b} = (b_1, \dots, b_m)$, and put $B' = B \otimes_R R'$. Write ϖ' and b'_i for the images of ϖ and b_i . Then:*

(i) *the algebraic analytic rings satisfy a canonical base-change equivalence*

$$B_R^{\text{alg}} \otimes_{(R, R^+)^{\square}}^{\text{AnRing}} (R', R'^+)^{\square} \simeq (B')_{R'}^{\text{alg}};$$

- (ii) *for every $n \geq 0$, the simultaneous bounded chart defined by $|\varpi^n b_i| \leq 1$ base changes to the chart defined by $|\varpi^n b'_i| \leq 1$, compatibly with the transition maps in n ;*
- (iii) *the increasing union of these charts is $(\text{Spec } B)^{\text{An}}$, and the equivalences in (ii) induce $(\text{Spec } B)^{\text{An}} \times_{\text{Spa}(R, R^+)} \text{Spa}(R', R'^+) \simeq (\text{Spec } B')^{\text{An}}$. These constructions are functorial in morphisms of affine schemes and compatible with a second rational localization. Replacing ϖ or $\mathbf{b}$ gives the same map after passage to the analytic-locus union.*

Proof. Put

$$\begin{aligned} \mathcal{R}_0 &= (R_{\text{disc}}, R_{\text{disc}}^+)^{\square}, \\ \mathcal{B}_0 &= (B_{\text{disc}}, B_{\text{disc}}^+)^{\square}, \end{aligned}$$

and write $\mathcal{R} = (R, R^+)^{\square}$. Use primes for the corresponding rings after base change. By definition, $B_R^{\text{alg}} = \mathcal{R} \otimes_{\mathcal{R}_0}^{\text{AnRing}} \mathcal{B}_0$ [LH26, A.9]. Lemma B.8 gives

$$\begin{aligned} B_R^{\text{alg}} \otimes_{\mathcal{R}}^{\text{AnRing}} \mathcal{R}' &\simeq \mathcal{R}' \otimes_{\mathcal{R}'_0}^{\text{AnRing}} \left(\mathcal{R}'_0 \otimes_{\mathcal{R}_0}^{\text{AnRing}} \mathcal{B}_0 \right) \\ &\simeq \mathcal{R}' \otimes_{\mathcal{R}'_0}^{\text{AnRing}} \mathcal{B}'_0 = (B')_{R'}^{\text{alg}}. \end{aligned}$$

This proves (i).

For the bounded chart, take the pushout

$$\begin{array}{ccc} (\mathbf{Z}[T_1, \dots, T_m], \mathbf{Z})^{\square} & \longrightarrow & \mathbf{Z}[T_1, \dots, T_m]^{\square} \\ \downarrow T_i \mapsto \varpi^n b_i & & \downarrow \\ B_R^{\text{alg}} & \longrightarrow & B_{|\varpi^n \mathbf{b}| \leq 1}^{\text{univ}}. \end{array}$$

The upper arrow imposes the integral bounds on the T_i . A map from the lower right corner to an analytic ring $\mathcal{C}$ is a map from B_R^{alg} for which each $\varpi^n b_i$ satisfies this universal boundedness condition. The upper map is a steady localization, so the chart is a subobject of $\text{AnSpec } B_R^{\text{alg}}$, and the factorization is unique up to contractible choice [LH26, A.9].

The upper row is independent of the base. Taking its pushout over $\mathcal{R}'$ and using (i) gives

$$B_{|\varpi^n \mathbf{b}| \leq 1}^{\text{univ}} \otimes_{\mathcal{R}}^{\text{AnRing}} \mathcal{R}' \simeq (B')_{|\varpi^n \mathbf{b}'| \leq 1}^{\text{univ}}.$$

Since $\varpi \in R^+$, the identity $\varpi^{n+1} b_i = \varpi(\varpi^n b_i)$ gives an inclusion of the n th chart into the $(n+1)$ st. The same identity holds over R' , so the base-change maps commute with these inclusions. This proves (ii).

The analytic locus is the union

$$(\text{Spec } B)^{\text{An}} \simeq \text{colim}_{n \geq 0} \text{AnSpec } B_{|\varpi^n \mathbf{b}| \leq 1}^{\text{univ}}$$

in analytic stacks [LH26, A.11–A.15]; the diagonal $\mathbf{N} \rightarrow \mathbf{N}^m$ is cofinal in the system with separate bounds for each generator. Pullback in the slice topos preserves colimits. Part (iii)

follows by taking the union of the chart comparisons. For successive rational localizations, the comparison maps are the associativity maps of the same iterated pushout.

We check independence of the choices. Write

$$U_{\varpi,n}(\mathbf{b}) = \text{AnSpec } B_{|\varpi^n \mathbf{b}| \leq 1}^{\text{univ}}.$$

Every topologically nilpotent element belongs to R^+ : a sufficiently large power lies in the open subring R^+ , and integral closedness gives the element itself. The universal boundedness maps for addition and multiplication show that sums and products of bounded elements are bounded [LH26, A.9 and the proof of Proposition A.12(2)].

Given two generating tuples, enlarge both to their union $\mathbf{c}$. Express an added generator as

$$c_j = \sum_{\nu} a_{j\nu} \mathbf{b}^{\nu}.$$

For fixed n , choose $m \geq n$ such that $\varpi^{m-n|\nu|} a_{j\nu} \in R^+$ for all the finitely many coefficients. This is possible because R^+ is open. On $U_{\varpi,n}(\mathbf{b})$, we have

$$\varpi^m c_j = \sum_{\nu} (\varpi^{m-n|\nu|} a_{j\nu}) (\varpi^n \mathbf{b})^{\nu}.$$

The universal addition and multiplication maps make the right side bounded. The boundedness chart pushout therefore gives an inclusion

$$U_{\varpi,n}(\mathbf{b}) \subset U_{\varpi,m}(\mathbf{c}).$$

Forgetting the added bounds gives $U_{\varpi,n}(\mathbf{c}) \subset U_{\varpi,n}(\mathbf{b})$. We may enlarge m monotonically with n , so the two systems have cofinal common refinements.

For a second topologically nilpotent unit $\tilde{\varpi}$, choose positive integers r, s such that

$$\frac{\tilde{\varpi}^r}{\varpi} \in R^+, \\ \frac{\tilde{\varpi}^s}{\varpi} \in R^+.$$

Again this follows from openness of R^+ . The identity

$$\tilde{\varpi}^{rn} c_i = \left(\frac{\tilde{\varpi}^r}{\varpi} \right)^n (\varpi^n c_i)$$

and its counterpart with the units interchanged give

$$U_{\varpi,n}(\mathbf{c}) \subset U_{\tilde{\varpi},rn}(\mathbf{c}), \\ U_{\tilde{\varpi},n}(\mathbf{c}) \subset U_{\varpi,sn}(\mathbf{c}).$$

These are morphisms of analytic stacks furnished by the same universal boundedness construction. They compare the two unions, and the base-change maps agree on every common refinement.

Finally, for an R -algebra map $B \rightarrow C$, express the images of the b_i as polynomials in generators of C . The preceding coefficient bounds make each bounded chart for C map to a sufficiently large one for B . Taking unions defines the map of analytic loci. Common refinements identify different choices. Since the maps are factorizations through localization subobjects, their spaces of choices are contractible; hence they respect composition and rational base change with all the required coherences. $\square$

B.4. Comparison over a Tate affinoid. For a complete Tate Huber pair, the functor

$$\mathrm{Cond}_R(M) = \underline{R} \otimes_{R_{\mathrm{disc}}}^{\mathbf{L}} M_{\mathrm{disc}}$$

is defined on all of $\mathrm{D}(R)$ and takes values in complete modules [And21, Theorem 5.9]. We apply it to modules over an R -algebra.

Lemma B.10. *Let (R, R^+) be a complete Tate Huber pair and let A be a finitely presented R -algebra. Write $(\mathrm{Spec} A)^{\mathrm{Alg}}$ for the algebraic realization over (R, R^+) defined in [LH26, A.9] and $q_A^{\mathrm{Alg}} : (\mathrm{Spec} A)^{\mathrm{Alg}} \rightarrow \mathrm{AnSpec}(R, R^+)^{\square}$ for its structural morphism. There is a canonical symmetric monoidal equivalence*

$$\mathcal{C}_{\square}((\mathrm{Spec} A)^{\mathrm{Alg}}) \simeq \mathrm{Mod}_{\mathrm{Cond}_R(A)}(\mathrm{D}_{\square}(R, R^+)).$$

Under this equivalence, $\mathbf{R}q_{A,*}^{\mathrm{Alg}}$ is the forgetful functor. Condensification $\mathrm{Cond}_R : \mathrm{D}(R) \rightarrow \mathrm{D}_{\square}(R, R^+)$ induces the comparison functor on modules. Consequently $\alpha_A^{\mathrm{Alg}} : \mathrm{D}(A) \rightarrow \mathcal{C}_{\square}((\mathrm{Spec} A)^{\mathrm{Alg}})$ is fully faithful. Its essential image consists exactly of the modules whose underlying object of $\mathrm{D}_{\square}(R, R^+)$ belongs to the essential image of Cond_R .

Proof. Let A^+ be the integral closure of the image of R^+ in A . Put

$$\begin{aligned} \mathcal{R}_0 &= (R_{\mathrm{disc}}, R_{\mathrm{disc}}^+)^{\square}, \\ \mathcal{A}_0 &= (A_{\mathrm{disc}}, A_{\mathrm{disc}}^+)^{\square}, \end{aligned}$$

and $\mathcal{R} = (R, R^+)^{\square}$. The definition [LH26, A.9] and the pushout formula for module categories [LH26, Lemma A.1] give

$$\begin{aligned} \mathrm{D}(A_R^{\mathrm{alg}}) &\simeq \mathrm{D}(\mathcal{R}) \otimes_{\mathrm{D}(\mathcal{R}_0)} \mathrm{D}(\mathcal{A}_0) \\ &\simeq \mathrm{D}(\mathcal{R}) \otimes_{\mathrm{D}(\mathcal{R}_0)} \mathrm{Mod}_A(\mathrm{D}(\mathcal{R}_0)) \\ &\simeq \mathrm{Mod}_{\mathrm{Cond}_R(A)}(\mathrm{D}_{\square}(R, R^+)). \end{aligned}$$

The second equivalence is Lemma B.7. The third is scalar extension of a module category [Lur17, Section 4.8.5]; the algebra A maps to the complete algebra $\mathrm{Cond}_R(A)$.

Under this description, pullback is the free-module functor and its right adjoint is the forgetful functor. The comparison is induced by Cond_R , which is fully faithful, exact, strong symmetric monoidal, and preserves colimits [And21, Theorem 5.9 and Lemma 5.10]. Lemma B.3 therefore gives full faithfulness and the essential-image criterion. $\square$

Proposition B.11. *Let (R, R^+) be a complete Tate Huber pair and let S be a separated finitely presented R -scheme. Scalar extension over the Tate base induces a fully faithful colimit-preserving strong symmetric monoidal functor $\alpha_S^{\mathrm{Alg}} : \mathrm{D}_{\mathrm{qc}}(S) \rightarrow \mathcal{C}_{\square}(S^{\mathrm{Alg}})$. It is functorial in S and compatible with pullback.*

Proof. Choose a finite affine Zariski cover (S_a) of S . Its multiple intersections are affine. On each intersection, Lemma B.10 gives the comparison functor, and scalar-extension compatibility of condensification identifies its restrictions [And21, Lemma 5.10]. Tensor associativity gives the compatibility for compositions.

Zariski descent computes $\mathrm{D}_{\mathrm{qc}}(S)$ from the source categories. The algebraic realization is the corresponding simplicial gluing [LH26, Theorem A.13], so Proposition B.1 computes its coefficient category from the target categories. Taking limits and applying Lemma B.2 proves full faithfulness and preservation of colimits. The tensor comparisons descend through the same diagram.

A common refinement compares two choices of cover. For a morphism of schemes, refine the source cover so that every affine maps into a target affine. Scalar extension then gives pullback compatibility, with composition governed by tensor associativity. $\square$

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