

ON THE SMOOTH TRANSFER OF ORBITAL INTEGRALS FOLLOWING WALDSPURGER

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ENGLISH TRANSLATION

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1. INTRODUCTION

This article is intended as an introduction to Waldspurger's article [9], which proves the existence of transfer assuming the fundamental lemma for Lie algebras, now a theorem. In fact, the existence of transfer is a local problem near semisimple elements. Descent arguments show that it is enough to solve the same problem for the Lie algebra. Waldspurger's approach uses an analogue, for Lie algebras, of the elliptic geometric part of the Arthur–Selberg trace formula. Its spectral counterpart is the Fourier transform of the geometric side, the two being related by the Poisson summation formula. One can then apply the stabilization techniques of Kottwitz and Langlands to compare these trace formulas on the Lie algebras of different groups.

Let us describe the contents of this article in more detail. Section 2 contains the notation used throughout. We recall the notions of abelianized cohomology, for which we refer to [5]. Endoscopic data are introduced in Section 3. We have undertaken the Oulipian exercise, dear to Labesse, of avoiding any mention of the Langlands dual group; this accounts for the unusual appearance of the definitions. Unless I am mistaken, their formulation should be equivalent to the usual definition. Section 4 explains how conjugacy classes transfer between a group and its endoscopic group, with particular emphasis on the Kostant section. Sections 5 and 6 recall some facts about p -adic harmonic analysis on

Lie algebras. In Section 7, we give a definition of transfer factors inspired by Kottwitz's result in [4]. The main statements of endoscopic theory for p -adic Lie algebras appear in Section 8: smooth transfer of orbital integrals, compatibility of transfer with Fourier transformation, and the fundamental lemma. The point of Waldspurger's work is to show that each statement follows from the next. Section 9 reviews the analogous theory for complex Lie algebras, whose results rest on those of Bouaziz. Pre-stabilization in our setting is reviewed in Section 10. In the final Section 11, we explain some points in Waldspurger's proof that "the fundamental lemma implies that transfer commutes with the Fourier transform." Consequently, as the title of his article puts it, "the fundamental lemma implies transfer."

2. NOTATION

2.1. Let F be a field of characteristic 0. We assume that F is either a p -adic field, that is, a finite extension of $\mathbb{Q}_p$, or a number field. Let $\bar{F}$ be an algebraic closure of F , and let $\Gamma = \text{Gal}(\bar{F}/F)$ be the associated Galois group.

2.2. Let G be a connected reductive group over F , and let $\mathfrak{g}$ be its Lie algebra. Let $G_{\text{der}} \subset G$ be its derived subgroup and G_{sc} the simply connected cover of G_{der} . Let Z_G be the center of G and $G_{\text{ad}} = G/Z_G$ the adjoint group. For any maximal subtorus T of G , let $\text{Norm}_G(T)$ be its normalizer in G and $W^G(T) = \text{Norm}_G(T)/T$ the Weyl group. Let $T_{G_{\text{sc}}}$ be the inverse image of T in G_{sc} , and set $T_{G_{\text{ad}}} = T/Z_G$. When the context is clear, we write $T_{\text{sc}} = T_{G_{\text{sc}}}$ and $T_{\text{ad}} = T_{G_{\text{ad}}}$. We will sometimes abuse notation by writing simply G instead of $G(\bar{F})$ for the group of $\bar{F}$ -points of G .

2.3. The group G acts on its Lie algebra by the adjoint action. Let $X \in \mathfrak{g}(F)$. By definition, X is *regular* if its centralizer G_X in G has minimal dimension, equal to the rank of G . If X is regular semisimple, its centralizer G_X is a torus. We then write $T_X = G_X$; this is a maximal subtorus of G defined over F .

Let $\mathfrak{g}_{\text{reg}}(F)$ be the set of regular semisimple elements of $\mathfrak{g}(F)$. We introduce the set of regular semisimple conjugacy classes

$$\Gamma(\mathfrak{g}) = \mathfrak{g}_{\text{reg}}(F)/G(F).$$

Its stable analogue is

$$\Sigma(\mathfrak{g}) = \mathfrak{g}_{\text{reg}}(F)/G(\bar{F}),$$

namely the quotient of $\mathfrak{g}_{\text{reg}}(F)$ by the following equivalence relation: two elements $X, Y \in \mathfrak{g}_{\text{reg}}(F)$ define the same class in $\Sigma(\mathfrak{g})$ if and only if there is $g \in G$ such that $X = \text{Ad}(g)Y$. We also say that X and Y are *stably conjugate*.

2.4. Let A_G be the maximal central split subtorus of G . For any torus T , the torus A_T is its maximal split subtorus.

Write $\mathfrak{g}_{\text{ell}}(F) \subset \mathfrak{g}_{\text{reg}}(F)$ for the subset of *elliptic* elements: these are the elements $X \in \mathfrak{g}_{\text{reg}}(F)$ whose centralizer T is elliptic, that is, satisfies either of the following equivalent conditions:

- $A_T = A_G$;
- $X_*(T_{G_{\text{sc}}})^\Gamma = 1$.

This notion of ellipticity is preserved by conjugacy and stable conjugacy. The sets $\Gamma_{\text{ell}}(\mathfrak{g})$ and $\Sigma_{\text{ell}}(\mathfrak{g})$ are then defined in the evident way.

2.5. In what follows, we abuse notation by not distinguishing a variety defined over F from the variety over $\bar{F}$ obtained by extending scalars. The latter is equipped with an action of Γ .

Let G^* be a quasi-split inner form of G . In other words, G^* is a connected quasi-split reductive group over F , and there is an isomorphism over $\bar{F}$

$$\varphi : G \longrightarrow G^* \quad (2.1)$$

such that $\varphi\sigma(\varphi)^{-1}$ is an inner automorphism of G^* for every $\sigma \in \Gamma$. In particular, there is a cocycle u with values in G_{ad}^* such that

$$\varphi\sigma(\varphi)^{-1} = \text{Ad}(u_\sigma)$$

for every $\sigma \in \Gamma$. The class of u in $H^1(F, G_{\text{ad}}^*)$ does not depend on the choice of φ . Moreover, up to F -isomorphism, the groups G that are inner forms of G^* are in natural bijection with $H^1(F, G_{\text{ad}}^*)$. The group G is quasi-split if and only if the class of the cocycle u is trivial. The group G^* is defined up to F -isomorphism.

2.6. Let T be a maximal subtorus of G defined over F . The short exact sequence

$$0 \longrightarrow T \longrightarrow G \longrightarrow T \backslash G \longrightarrow 0$$

induces a long exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^0(F, T) \longrightarrow H^0(F, G) \longrightarrow H^0(F, T \backslash G) \\ \longrightarrow H^1(F, T) \longrightarrow H^1(F, G). \end{aligned}$$

Following Langlands, Kottwitz, and Labesse, set

$$\begin{aligned} \mathcal{D}(T, G, F) &= \ker\left(H^1(F, T) \longrightarrow H^1(F, G)\right) \\ &= \text{coker}\left(H^0(F, G) \longrightarrow H^0(F, T \backslash G)\right). \end{aligned}$$

Bear in mind that $H^1(F, G)$ is not a group, but only a set pointed by the class of the trivial cocycle. Let $X \in \mathfrak{g}(F)$ be regular semisimple. The set $\mathcal{D}(T_X, G, F)$ is in bijection with the set of conjugacy classes in the stable conjugacy class of X . Recall how this bijection is obtained. Let $Y \in \mathfrak{g}(F)$ be stably conjugate to X . There is $g \in G(\bar{F})$ such that $Y = \text{Ad}(g)X$. For every $\sigma \in \Gamma$, one checks that $g^{-1}\sigma(g)$ centralizes X , and hence belongs to T_X . This gives an element of $\mathcal{D}(T_X, G, F)$.

The following long exact sequence is an “abelianized” version of the preceding one:

$$\begin{aligned} H^0(F, T) \longrightarrow H^0(F, T_{\text{sc}} \rightarrow T) \longrightarrow H^1(F, T_{\text{sc}}) \\ \longrightarrow H^1(F, T) \longrightarrow H^1(F, T_{\text{sc}} \rightarrow T). \end{aligned}$$

Thus we define

$$\begin{aligned} \mathcal{E}(T, G, F) &= \ker\left(H^1(F, T) \longrightarrow H^1(F, T_{\text{sc}} \rightarrow T)\right) \\ &= \text{coker}\left(H^0(F, T_{\text{sc}} \rightarrow T) \longrightarrow H^1(F, T_{\text{sc}})\right). \end{aligned}$$

Recall that Labesse defined cohomology sets $H^i(F, G_{\text{sc}} \rightarrow G)$ for the crossed module $[G_{\text{sc}} \rightarrow G]$, and that these are isomorphic to $H^i(F, T_{\text{sc}} \rightarrow T)$. In degrees $i = 0, 1$, there are abelianization maps

$$H^i(F, G) \longrightarrow H^i(F, G_{\text{sc}} \rightarrow G),$$

and hence a map

$$\mathcal{D}(T, G, F) \hookrightarrow \mathcal{E}(T, G, F),$$

which is clearly injective. It is also bijective when F is nonarchimedean local. Indeed, in that case the abelianization map

$$H^1(F, G) \longrightarrow H^1(F, G_{\text{sc}} \rightarrow G)$$

is injective: in general, its kernel is the image of $H^1(F, G_{\text{sc}}) \rightarrow H^1(F, G)$, but when F is nonarchimedean local it is well known that

$$H^1(F, G_{\text{sc}}) = 1.$$

If F is a number field, we write $\mathbb{A}_F/F$ for the ring of adeles of F and set

$$\mathcal{E}(T, G, \mathbb{A}_F/F) = \text{coker}\left(H^0(\mathbb{A}_F, T_{\text{sc}} \rightarrow T) \longrightarrow H^1(\mathbb{A}_F/F, T_{\text{sc}})\right).$$

Let C_F be the field F when F is local, or the F -algebra of adele classes $\mathbb{A}_F/F$ when F is global. In both cases we define the *group of endoscopic characters*

$$\mathcal{K}(T, G, F) = \mathcal{E}(T, G, C_F)^D,$$

where the superscript D denotes Pontryagin duality.

3. ENDOSCOPIC DATA

3.1. The definition. Let G be as in the preceding section. An *endoscopic datum* for G is a triple (H, κ, ξ) , where

- H is a connected quasi-split reductive group over F ;
- ξ is a commutative diagram

$$\xi : \begin{array}{ccc} T_{H_{\text{sc}}} & \hookrightarrow & T_{G_{\text{sc}}} \\ \downarrow & & \downarrow \\ T_H & \xrightarrow{\iota} & T_G. \end{array} \quad (3.1)$$

Here T_H and T_G are maximal subtori defined over F of H and G , respectively; the vertical arrows are the evident maps, the horizontal arrows are F -embeddings, and ι is an F -isomorphism;

- κ is a character of $H_0(F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}}))$, where $X_*(T_{H_{\text{sc}}})$ is identified with its image in $X_*(T_{G_{\text{sc}}})$.

These objects must satisfy two conditions, which we now spell out. Let R_G and $R_G^\vee$ be the sets of roots and coroots of T in G , respectively. Note that $R_G^\vee$ identifies with a subset of the submodule $X_*(T_{G_{\text{sc}}})$ of $X_*(T)$. Set

$$R_\kappa^\vee = \{\alpha^\vee \in R_G^\vee \mid \langle \kappa, \alpha^\vee \rangle = 1\}, \quad R_\kappa = \{\alpha \in R_G \mid \alpha^\vee \in R_\kappa^\vee\},$$

where $\alpha^\vee$ is the coroot of α .

The first condition is as follows. The isomorphism ι in (3.1) induces an F -isomorphism of root data

$$\begin{aligned} & (X^*(T_H), R_H, X_*(T_H), R_H^\vee) \\ & \simeq (X^*(T_G), R_\kappa, X_*(T_G), R_\kappa^\vee). \end{aligned} \quad (3.2)$$

To express the second condition, fix a Galois extension E of F that splits the tori $T_{G_{\text{sc}}}$ and $T_{H_{\text{sc}}}$. We can then regard κ as a character of $H_0(E/F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}}))$. Composing the evident map

$$\begin{aligned} & \widehat{H}^{-1}(E/F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}})) \\ & \longrightarrow H_0(E/F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}})) \end{aligned}$$

with the Tate–Nakayama isomorphism

$$H^1(C_F, T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}}) \simeq \widehat{H}^{-1}(E/F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}})) ,$$

we see that κ induces a character on $H^1(C_F, T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}})$, and hence a character κ_0 on $H^1(C_F, T_{G_{\text{sc}}})$ through the map

$$H^1(C_F, T_{G_{\text{sc}}}) \longrightarrow H^1(C_F, T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}}).$$

The second condition concerns this character κ_0 . We require that κ_0 belong to $\mathcal{K}(T_G, G, F)$, that is, that it be trivial on the image of $H^0(\mathbb{A}_F, T_{G_{\text{sc}}} \rightarrow T_G)$ if F is global, or of $H^0(F, T_{G_{\text{sc}}} \rightarrow T_G)$ if F is local.

3.2. Equivalence. Let (H, κ, ξ) and (H', κ', ξ') be two endoscopic data for G . Thus we have two commutative diagrams

$$\begin{array}{ccc} \xi : & \begin{array}{ccc} T_{H_{\text{sc}}} & \hookrightarrow & T_{G_{\text{sc}}} \\ \downarrow & & \downarrow \\ T_H & \xrightarrow{\iota} & T_G \end{array} & \text{and} & \xi' : \begin{array}{ccc} T_{H'_{\text{sc}}} & \hookrightarrow & T'_{G_{\text{sc}}} \\ \downarrow & & \downarrow \\ T_{H'} & \xrightarrow{\iota} & T'_G \end{array} \end{array}$$

By definition, these data are *equivalent* if there is an F -isomorphism $\alpha : H' \rightarrow H$ and elements $h \in H_{\text{sc}}$ and $g \in G_{\text{sc}}$ satisfying the following two properties.

- (1) We have $(\text{Ad}(h) \circ \alpha)(T_{H'}) = T_H$ and $\text{Ad}(g)(T'_G) = T_G$. The maps $\text{Ad}(h) \circ \alpha$ and $\text{Ad}(g)$ induce an isomorphism over $\overline{F}$ from ξ' to ξ . In particular, we have commutative diagrams

$$\begin{array}{ccc} T_{H'_{\text{sc}}} & \longrightarrow & T'_{G_{\text{sc}}} \\ \text{Ad}(h) \circ \alpha_{\text{sc}} \downarrow & & \downarrow \text{Ad}(g) \\ T_{H_{\text{sc}}} & \longrightarrow & T_{G_{\text{sc}}} \end{array}$$

where α_{sc} is the unique isomorphism $H'_{\text{sc}} \rightarrow H_{\text{sc}}$ lifting α , and

$$\begin{array}{ccc} T_{H'} & \longrightarrow & T'_G \\ \text{Ad}(h) \circ \alpha \downarrow & & \downarrow \text{Ad}(g) \\ T_H & \longrightarrow & T_G, \end{array}$$

compatible in the evident sense with the diagrams ξ and ξ' . The first diagram induces an isomorphism

$$X_*(T_{H'_{\text{sc}}}) \backslash X_*(T'_{G_{\text{sc}}}) \longrightarrow X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}}). \quad (3.3)$$

Let us show that it is equivariant. It is enough to prove the analogous assertion with $T'_{G_{\text{sc}}}$ and $T_{G_{\text{sc}}}$ replaced by T'_G and T_G . Since the latter are F -isomorphic to $T_{H'}$ and T_H , it suffices to show that the isomorphism

$$X_*(T_{H'_{\text{sc}}}) \backslash X_*(T_{H'}) \longrightarrow X_*(T_{H_{\text{sc}}}) \backslash X_*(T_H)$$

induced by $\text{Ad}(h) \circ \alpha$ is Γ -equivariant. Now

$$\text{Ad}(h) \circ \alpha \circ \sigma(\text{Ad}(h) \circ \alpha)^{-1} = \text{Ad}(h\sigma(h)^{-1}),$$

and $h\sigma(h)^{-1}$ belongs to the normalizer of T_H in H . It therefore suffices to check that every element of $W^H(T_H)$ acts trivially on $X_*(T_{H_{\text{sc}}}) \backslash X_*(T_H)$. It is enough to check this on the generating set of simple reflections s_α , where α ranges over the roots of T_H in H . This is immediate from the formula, valid for every $\chi \in X_*(T_H)$,

$$s_\alpha(\chi) - \chi = \alpha(\chi)\alpha^\vee,$$

since $\alpha^\vee \in X_*(T_{H_{\text{sc}}})$.

(2) Identify κ' with a character of $H_0(F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}}))$ by means of the isomorphism

$$\begin{aligned} & H_0(F, X_*(T_{H'_{\text{sc}}}) \backslash X_*(T'_{G_{\text{sc}}})) \\ & \longrightarrow H_0(F, X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}})) \end{aligned}$$

induced by (3.3). Let κ'_0 be the character of $H^1(C_F, T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}})$ induced by κ' . One checks that κ'_0 belongs to $\mathcal{K}(T_G, G, F)$. Let κ_0 be the element of $\mathcal{K}(T_G, G, F)$ induced by κ . The second condition is the equality $\kappa_0 = \kappa'_0$.

3.3. Ellipticity. An endoscopic datum (H, κ, ξ) for G is *elliptic* if $\iota(A_H) = A_G$; the inclusion $\supset$ always holds.

4. TRANSFER OF CONJUGACY CLASSES

4.1. Let $\overline{F}[\mathfrak{g}]$ be the $\overline{F}$ -algebra of regular polynomial functions on $\mathfrak{g}$. It has an evident action of Γ . Let $\overline{F}[\mathfrak{g}]^G$ be the subalgebra of G -invariant functions: it is stable under Γ and is a polynomial algebra over $\overline{F}$. The affine space $\mathfrak{c}_G = \text{Spec}(\overline{F}[\mathfrak{g}]^G)$ also has an action of Γ . Let

$$\chi = \chi_G : \mathfrak{g} \longrightarrow \mathfrak{c}_G$$

be the morphism dual to the inclusion $\overline{F}[\mathfrak{g}]^G \subset \overline{F}[\mathfrak{g}]$. When $X \in \mathfrak{g}$ is regular semisimple, the fiber $\chi^{-1}(\chi(X))$ is the G -conjugacy class of X . Note that χ is defined over F . When $X \in \mathfrak{g}_{\text{reg}}(F)$, the F -points of the fiber $\chi^{-1}(\chi(X))$ form the stable conjugacy class of X .

The map χ induces maps, still denoted by χ , from $\Gamma(\mathfrak{g})$ and $\Sigma(\mathfrak{g})$ to $\mathfrak{c}_G$.

4.2. Let T be a maximal subtorus of G and $W = W^G(T)$. The restriction map $\overline{F}[\mathfrak{g}] \rightarrow \overline{F}[\mathfrak{t}]$ induces the Chevalley isomorphism

$$\overline{F}[\mathfrak{g}]^G \longrightarrow \overline{F}[\mathfrak{t}]^W$$

onto the algebra of W -invariant polynomials on $\mathfrak{t}$. If T is defined over F , this isomorphism is Γ -equivariant.

4.3. Transfer of conjugacy classes. Let (H, κ, ξ) be an endoscopic datum for G . Let T_H and T_G be its underlying maximal F -subtori of H and G , and let $\iota : T_H \rightarrow T_G$ be its underlying F -isomorphism. The definition of an endoscopic datum implies that $W^H = W^H(T_H)$ is naturally a subgroup of $W^G = W^G(T_G)$ and that ι is W^H -equivariant. In particular, ι induces a Γ -equivariant map

$$\overline{F}[\mathfrak{t}_G]^{W^G} \longrightarrow \overline{F}[\mathfrak{t}_H]^{W^H}.$$

Combining this map with the Chevalley isomorphism, we obtain a Γ -equivariant morphism of affine schemes

$$\mathfrak{c}_H \longrightarrow \mathfrak{c}_G. \quad (4.1)$$

Let $Y \in \mathfrak{h}$ and $X \in \mathfrak{g}$. We say that Y is an *image* of X , and write $Y \rightarrow X$, if $\chi_H(Y)$ maps to $\chi_G(X)$ under this morphism. We say that Y is *G -regular* if there exists a regular semisimple X such that $Y \rightarrow X$, or equivalently if every X such that $Y \rightarrow X$ is regular semisimple.

Definition 4.1. Let $Y \in \mathfrak{h}(F)$ be G -regular. We say that Y *transfers to $\mathfrak{g}$* if there is $X \in \mathfrak{g}_{\text{reg}}(F)$ such that $Y \rightarrow X$.

If $Y \rightarrow X$, the same holds for any $Y' \in \mathfrak{h}(F)$ and $X' \in \mathfrak{g}(F)$ such that Y' is H -stably conjugate to Y and X' is G -stably conjugate to X . All $X' \in \mathfrak{g}(F)$ such that $Y \rightarrow X'$ are stably conjugate. Let $\Sigma_G(\mathfrak{h}) \subset \Sigma(\mathfrak{h})$ be the subset of G -regular classes. We have thus defined a function

$$\Sigma_G(\mathfrak{h}) \longrightarrow \Sigma(\mathfrak{g}),$$

whose domain consists of the classes that are images of elements of $\mathfrak{g}_{\text{reg}}(F)$, and whose fibers are readily checked to be finite.

4.4. Let $Y \in \mathfrak{h}(F)$ be G -regular and $X \in \mathfrak{g}_{\text{reg}}(F)$. One checks that $Y \rightarrow X$ if and only if there are $h \in H_{\text{sc}}$ and $g \in G_{\text{sc}}$ such that

- $\text{Ad}(h)T_Y = T_H$;
- $\text{Ad}(g)T_G = T_X$;
- the composite morphism

$$T_Y \xrightarrow{\text{Ad}(h)} T_H \xrightarrow{\iota} T_G \xrightarrow{\text{Ad}(g)} T_X \quad (4.2)$$

is an F -isomorphism carrying Y to X .

The isomorphism (4.2) extends in the evident way to a diagram

$$\xi' : \begin{array}{ccc} T_{Y, H_{\text{sc}}} & \hookrightarrow & T_{X, G_{\text{sc}}} \\ \downarrow & & \downarrow \\ T_Y & \longrightarrow & T_X. \end{array}$$

The inner automorphisms $\text{Ad}(h)$ and $\text{Ad}(g^{-1})$ induce an isomorphism from ξ' to ξ . As we saw after (3.3), this implies that $\text{Ad}(g^{-1})$ induces a Γ -equivariant isomorphism

$$X_*(T_{Y, H_{\text{sc}}}) \backslash X_*(T_{X, G_{\text{sc}}}) \longrightarrow X_*(T_{H_{\text{sc}}}) \backslash X_*(T_{G_{\text{sc}}}). \quad (4.3)$$

Through this isomorphism, κ defines a character κ' of $H_0(X_*(T_{Y,H_{sc}}) \backslash X_*(T_{X,G_{sc}}))$. One checks that κ' is endoscopic: it defines on $H^1(C_F, T_{X,G_{sc}})$ a character κ_Y belonging to $\mathcal{K}(T_X, G, F)$. We therefore have an endoscopic datum (H, κ', ξ') . The identity of H extends to an equivalence between (H, κ, ξ) and (H, κ', ξ') .

4.5. Two equivalent endoscopic data (H, κ, ξ) and (H, κ', ξ') give the same morphism $\mathfrak{c}_H \rightarrow \mathfrak{c}_G$ in (4.1), and the same function $\Sigma_G(\mathfrak{h}) \rightarrow \Sigma(\mathfrak{g})$.

4.6. In this paragraph, assume that $G = G^*$, that is, that G is quasi-split. Let B be a Borel subgroup of G defined over F , and let T be a subtorus of B defined over F . Let Φ be the set of simple roots of T in B . For every root α of T in G , the eigenspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \text{for every } t \in T, \text{Ad}(t)X = \alpha(t)X\}$$

is one-dimensional over $\overline{F}$. For each $\alpha \in \Phi$, fix a nonzero element $X_\alpha \in \mathfrak{g}_\alpha$ such that $\sigma(X_\alpha) = X_{\sigma(\alpha)}$ for every $\sigma \in \Gamma$. The triple $(B, T, \{X_\alpha\})$ is called a *pinning*.

For every $\alpha \in \Phi$, let $X_{-\alpha}$ be the unique element of $\mathfrak{g}_{-\alpha}$ characterized by the condition that $[X_\alpha, X_{-\alpha}]$ is the coroot $\alpha^\vee$, viewed as an element of $\mathfrak{t}$. Set

$$X_+ = \sum_{\alpha \in \Phi} X_\alpha, \quad X_- = \sum_{\alpha \in \Phi} X_{-\alpha}.$$

Let $\mathfrak{g}_{X_+}$ be the centralizer of X_+ in $\mathfrak{g}$. We state the following theorem of Kostant.

Theorem 4.2. *The restriction of χ to $X_- + \mathfrak{g}_{X_+}$ induces an isomorphism from $X_- + \mathfrak{g}_{X_+}$ onto $\mathfrak{c}$. In particular, it induces an isomorphism from $X_- + \mathfrak{g}_{X_+}(F)$ onto $\mathfrak{c}(F)$.*

We obtain the following corollaries.

Corollary 4.3. *Every stable class in $\Sigma(\mathfrak{g})$ contains exactly one element of $X_- + \mathfrak{g}_{X_+}$. The conjugacy class of this element does not depend on the choice of pinning.*

Corollary 4.4. *Let (H, κ, ξ) be an endoscopic datum for a quasi-split group G . Every G -regular element of $\mathfrak{h}(F)$ transfers to $\mathfrak{g}$.*

4.7. We no longer assume that G is quasi-split. The $\overline{F}$ -isomorphism φ in (2.1) induces an F -isomorphism $\mathfrak{c}_G \rightarrow \mathfrak{c}_{G^*}$. The preceding corollary leads us to study the conditions under which $X^* \in \mathfrak{g}_{\text{reg}}^*(F)$ transfers to $\mathfrak{g}$, that is, under which the fiber $\chi_G^{-1}(\chi_{G^*}(X^*))$ contains rational points.

Lemma 4.5. *Let $X^* \in \mathfrak{g}_{\text{reg}}^*(F)$. The following conditions are equivalent.*

1. X^* transfers to $\mathfrak{g}$.
2. There exist $g \in G_{\text{sc}}^*$ and $X \in \mathfrak{g}_{\text{reg}}(F)$ such that $\text{Ad}(g) \circ \varphi(X) = X^*$.
3. The class of φ in $H^1(F, G_{\text{ad}}^*)$ belongs to the image of

$$H^1(F, T_{X^*, \text{ad}}) \longrightarrow H^1(F, G_{\text{ad}}^*).$$

If F is nonarchimedean local, every elliptic element of $\mathfrak{g}^(F)$ transfers to $\mathfrak{g}$.*

Proof. Clearly, assertions 1 and 2 are equivalent, and 2 implies 3. We therefore only prove that 3 implies 2. To simplify notation, put $T = T_{X^*}$. Let u_σ be the cocycle with values in G_{ad}^* defined by $\varphi\sigma(\varphi)^{-1}$. Assertion 3 says that its class lies in the image of

$$H^1(F, T_{\text{ad}}) \longrightarrow H^1(F, G_{\text{ad}}^*),$$

that is, there is $g \in G_{\text{sc}}^*$ such that $gu_\sigma\sigma(g)^{-1}$ belongs to T/Z_{G^*} . Let $X \in \mathfrak{g}$ satisfy $\text{Ad}(g) \circ \varphi(X) = X^*$. One checks that $X \in \mathfrak{g}_{\text{reg}}(F)$.

We now prove the last assertion, assuming that F is nonarchimedean local. Since X^* is elliptic, T is elliptic, and the Tate–Nakayama isomorphism gives $H^2(F, T_{\text{sc}}) = 1$. We must prove that

$$H^1(F, T_{\text{ad}}) \longrightarrow H^1(F, G_{\text{ad}}^*)$$

is surjective. The exact sequence

$$H^1(F, T_{\text{ad}}) \longrightarrow H^1(F, T_{\text{sc}} \rightarrow T_{\text{ad}}) \longrightarrow H^2(F, T_{\text{sc}})$$

implies the surjectivity of $H^1(F, T_{\text{ad}}) \rightarrow H^1(F, T_{\text{sc}} \rightarrow T_{\text{ad}})$. The desired surjectivity now follows from the commutative diagram

$$\begin{array}{ccc} H^1(F, T_{\text{ad}}) & \longrightarrow & H^1(F, G_{\text{ad}}^*) \\ \downarrow & & \downarrow \\ H^1(F, T_{\text{sc}} \rightarrow T_{\text{ad}}) & \longrightarrow & H^1(F, G_{\text{sc}}^* \rightarrow G_{\text{ad}}^*) \end{array}$$

and the injectivity of the abelianization map, the right vertical arrow, over a nonarchimedean local field, since $H^1(F, G_{\text{sc}}^*) = 1$ over such a field. $\square$

4.8. The following lemma is an immediate consequence of the preceding corollary and lemma.

Lemma 4.6. *Let F be a nonarchimedean local field. Suppose that the endoscopic datum (H, κ, ξ) is elliptic. Then every elliptic G -regular element of $\mathfrak{h}(F)$ transfers to G .*

5. HARMONIC ANALYSIS ON p -ADIC LIE ALGEBRAS

5.1. Throughout this section, F is a p -adic field. We recall some results about harmonic analysis on $\mathfrak{g}(F)$ that will be useful later. An excellent reference is [2].

5.2. Let $C_c^\infty(\mathfrak{g}(F))$ be the vector space of smooth, that is, locally constant, compactly supported complex-valued functions on $\mathfrak{g}(F)$. Let $\mathcal{O}_F$ be the ring of integers of F . By a *lattice* in $\mathfrak{g}(F)$, we mean an $\mathcal{O}_F$ -submodule that is both open and compact. If R is such a lattice, write $C_c^\infty(\mathfrak{g}(F)/R)$ for the subspace of $C_c^\infty(\mathfrak{g}(F))$ consisting of R -invariant functions.

5.3. By a *distribution* on $\mathfrak{g}(F)$, we mean a linear form on $C_c^\infty(\mathfrak{g}(F))$. If T is a distribution, its support is denoted by $\text{supp}(T)$. The action of $G(F)$ on $\mathfrak{g}(F)$ defines, by duality, an action on $C_c^\infty(\mathfrak{g}(F))$ and on the space of distributions. A distribution is $G(F)$ -*invariant*, or simply *invariant*, if it is fixed by this action. Let J be the space of invariant distributions.

For subsets $\omega \subset \mathfrak{g}(F)$ and $H \subset G(F)$, set

$$\omega^H = \{\text{Ad}(x)X \mid x \in H \text{ and } X \in \omega\}.$$

For $\omega \subset \mathfrak{g}(F)$, let $J(\omega)$ be the space of invariant distributions T whose support is contained in the closure of $\omega^{G(F)}$.

Here is a first fundamental finiteness result, due to Howe and Harish-Chandra; it is Theorem 12.1 of [2].

Theorem 5.1. *Let $\omega \subset \mathfrak{g}(F)$ be compact and let R be a lattice in $\mathfrak{g}(F)$. Let $J_R(\omega)$ be the space spanned by the restrictions to $C_c^\infty(\mathfrak{g}(F)/R)$ of the elements of $J(\omega)$. Then*

$$\dim_{\mathbb{C}} J_R(\omega) < \infty.$$

5.4. Let

$$\langle \cdot, \cdot \rangle : \mathfrak{g}(F) \times \mathfrak{g}(F) \longrightarrow F$$

be a nondegenerate symmetric bilinear form invariant under $G(F)$. Let $\psi : F \rightarrow \mathbb{C}^\times$ be a continuous nontrivial character. Every subspace of $\mathfrak{g}(F)$ on which the restriction of this form is nondegenerate is equipped with the self-dual Haar measure for the bicharacter $\psi(\langle \cdot, \cdot \rangle)$.

We define a Fourier transformation, a linear bijection from $C_c^\infty(\mathfrak{g}(F))$ onto itself, by

$$\hat{f}(X) = \int_{\mathfrak{g}(F)} f(Y) \psi(\langle X, Y \rangle) dY.$$

By duality, we define a Fourier transform on the space of distributions. The action of $G(F)$ commutes with Fourier transformation. In particular, Fourier transformation preserves J .

5.5. We can now state Harish-Chandra's regularity theorem; see [2, Theorem 4.4].

Theorem 5.2. *Let $\omega \subset \mathfrak{g}(F)$ be compact and let $T \in J(\omega)$. There exists a function $\hat{t}$, locally constant on $\mathfrak{g}_{\text{reg}}(F)$ and locally integrable on $\mathfrak{g}(F)$, such that*

$$\hat{T}(f) = \int_{\mathfrak{g}(F)} \hat{t}(X) f(X) dX.$$

6. ORBITAL INTEGRALS OVER A p -ADIC FIELD

6.1. We retain the notation of the preceding section.

6.2. Fix a Haar measure on $G(F)$. Let $\mathcal{T}^G$ be the set of $G(F)$ -conjugacy classes of maximal F -subtori of G . In what follows, we identify this set with a set of representatives. Equip the tori in $\mathcal{T}^G$ with Haar measures. Every maximal F -subtorus of G is conjugate to a torus in $\mathcal{T}^G$; it is then equipped with the Haar measure obtained by transport.

6.3. Let $X \in \mathfrak{g}_{\text{reg}}(F)$. The *orbital integral* of X is the invariant distribution I_X defined by

$$I_X(f) = |D^{\mathfrak{g}}(X)|_F^{1/2} \int_{T_X(F) \backslash G(F)} f(\text{Ad}(x^{-1})X) dx,$$

where dx is the quotient Haar measure and $D^{\mathfrak{g}}(X) = \det(\text{ad}(X)|_{\mathfrak{g}/\mathfrak{t}_X})$ is the Weyl discriminant. The integral converges. This follows because the function

$$x \in T_X(F) \backslash G(F) \longmapsto f(\text{Ad}(x^{-1})X)$$

has compact support: indeed, $x \mapsto \text{Ad}(x^{-1})X$ induces a homeomorphism from $T_X(F) \backslash G(F)$ onto the orbit of X , and this orbit is closed because X is semisimple.

The importance of regular semisimple orbital integrals in harmonic analysis comes from the following theorem of Harish-Chandra: regular semisimple orbital integrals are weakly dense in J , in the sense that, if $I_X(f) = 0$ for every $X \in \mathfrak{g}_{\text{reg}}(F)$, then $T(f) = 0$ for every $T \in J$; see [2, Theorem 3.1].

6.4. A fundamental problem is to characterize the functions

$$X \in \mathfrak{g}_{\text{reg}}(F) \longmapsto I_X(f)$$

obtained as f ranges over $C_c^\infty(\mathfrak{g}(F))$. In general, this is a very difficult problem. When we restrict to $f \in C_c^\infty(\mathfrak{g}_{\text{reg}}(F))$, that is, smooth functions whose support is a compact subset of $\mathfrak{g}_{\text{reg}}(F)$, the problem is easy.

Lemma 6.1. *Let $I : \mathfrak{g}_{\text{reg}}(F) \rightarrow \mathbb{C}$. The following conditions are equivalent.*

1. *I is locally constant, invariant under $G(F)$, and has support contained in a set $\omega^{G(F)}$ with ω compact.*
2. *There is $f \in C_c^\infty(\mathfrak{g}_{\text{reg}}(F))$ such that*

$$I(X) = I_X(f) \quad \text{for every } X \in \mathfrak{g}_{\text{reg}}(F).$$

Proof. Let $T \in \mathcal{T}^G$ and consider

$$\begin{aligned} \Phi_T : T(F) \backslash G(F) \times \mathfrak{t}_{\text{reg}}(F) &\longrightarrow \mathfrak{g}_{\text{reg}}(F), \\ (g, X) &\longmapsto \text{Ad}(g^{-1})X. \end{aligned}$$

A little differential calculus shows that Φ_T induces an analytic covering of an open subset $\mathfrak{g}_T(F) \subset \mathfrak{g}_{\text{reg}}(F)$ by $T(F) \backslash G(F) \times \mathfrak{t}_{\text{reg}}(F)$; this covering is Galois with group $W^G(T)$. The open sets $\mathfrak{g}_T(F)$, for $T \in \mathcal{T}^G$, are pairwise disjoint and their union is $\mathfrak{g}_{\text{reg}}(F)$.

Let $f \in C_c^\infty(\mathfrak{g}_{\text{reg}}(F))$. We show that $X \mapsto I_X(f)$ satisfies condition 1. Only local constancy is not immediate. It is enough to show that, for every $T \in \mathcal{T}^G$, the function $(g, X) \mapsto I_{\Phi_T(g, X)}(f)$ is locally constant. Now $f \circ \Phi_T$ belongs to $C_c^\infty(T(F) \backslash G(F) \times \mathfrak{t}_{\text{reg}}(F))$. Thus there are finite families $\alpha_i \in C_c^\infty(T(F) \backslash G(F))$ and $\theta_i \in C_c^\infty(\mathfrak{t}_{\text{reg}}(F))$ such that

$$f \circ \Phi_T = \sum_i \alpha_i \otimes \theta_i.$$

For $(g, X) \in T(F) \backslash G(F) \times \mathfrak{t}_{\text{reg}}(F)$, we calculate

$$\begin{aligned} I_{\Phi_T(g, X)}(f) &= |D^\mathfrak{g}(X)|_F^{1/2} \int_{T_{\text{Ad}(g^{-1})X}(F) \backslash G(F)} f(\text{Ad}(x^{-1}g^{-1})X) dx \\ &= |D^\mathfrak{g}(X)|_F^{1/2} \int_{T_X(F) \backslash G(F)} f(\text{Ad}(x^{-1})X) dx \\ &= |D^\mathfrak{g}(X)|_F^{1/2} \sum_i \left(\int_{T_X(F) \backslash G(F)} \alpha_i(x) dx \right) \theta_i(X), \end{aligned}$$

and the result is now evident.

Conversely, suppose that I satisfies condition 1. Let $T \in \mathcal{T}^G$. The function $I \circ \Phi_T$ is constant in the first variable and defines a function $\theta \in C_c^\infty(\mathfrak{t}_{\text{reg}}(F))$ of the second variable. Its support is regular and compact because

$$\text{supp}(\theta) \subset \mathfrak{t}_{\text{reg}}(F) \cap \omega^{G(F)},$$

and the latter set is compact. Let $\alpha \in C_c^\infty(T(F) \backslash G(F))$ be W -invariant and satisfy

$$\int_{T_X(F) \backslash G(F)} \alpha(x) dx = 1.$$

Then $\alpha \otimes |D^\mathfrak{g}(\cdot)|_F^{-1/2} \theta$ is W -invariant and belongs to $C_c^\infty(T(F) \backslash G(F) \times \mathfrak{t}_{\text{reg}}(F))$. It therefore defines $f_T \in C_c^\infty(\mathfrak{g}_T(F))$. Letting T vary, we obtain a function f satisfying condition 2. $\square$

The following lemma is then immediate.

Lemma 6.2. *Let $f \in C_c^\infty(\mathfrak{g}_{\text{reg}}(F))$. The function $X \mapsto I_X(f)$ satisfies the assertions of the preceding lemma.*

6.5. Here is a more precise and slightly different formulation of Harish-Chandra's regularity theorem, Theorem 5.2, for orbital integrals.

Theorem 6.3. *There is a unique locally constant $G(F) \times G(F)$ -invariant function*

$$\hat{i}^G : \mathfrak{g}_{\text{reg}}(F) \times \mathfrak{g}_{\text{reg}}(F) \longrightarrow \mathbb{C}$$

such that, for every $X \in \mathfrak{g}_{\text{reg}}(F)$ and $f \in C_c^\infty(\mathfrak{g}(F))$,

$$\hat{I}_X(f) = \sum_{T \in \mathcal{T}^G} |W^G(T)|^{-1} \int_{\mathfrak{t}(F)} I_Y(f) \hat{i}^G(X, Y) dY.$$

The two formulations are related by the Weyl integration formula: there are constants c_T , for $T \in \mathcal{T}^G$, such that, for every $f \in C_c^\infty(\mathfrak{g}(F))$,

$$\begin{aligned} \int_{\mathfrak{g}(F)} f(X) dX &= \sum_{T \in \mathcal{T}^G} c_T |W^G(T)|^{-1} \int_{\mathfrak{t}(F)} |D^\mathfrak{g}(X)|_F \\ &\quad \cdot \int_{T(F) \backslash G(F)} f(\text{Ad}(x^{-1})X) dx dX. \end{aligned}$$

Local constancy of $\hat{i}^G$ in the first variable follows from Lemma 6.2.

Remark. It is clear that $\hat{i}^G$ does not depend on the choice of Haar measures on $G(F)$ and the tori $T(F)$.

7. TRANSFER FACTORS

7.1. In this section, F is a local field.

7.2. Let (H, κ, ξ) be an endoscopic datum for G . The purpose of this section is to define the Langlands–Shelstad *transfer factor*. It is a map

$$\Delta : \Sigma_G(\mathfrak{h}) \times \Gamma(\mathfrak{g}) \longrightarrow \mathbb{C}$$

such that $\Delta(Y, X) \neq 0$ if and only if $Y \rightarrow X$. We now specify its nonzero values. To do so, we define a canonical relative transfer factor $\Delta(Y_1, X_1, Y_2, X_2)$ for elements $Y_i \in \Sigma_G(\mathfrak{h})$ and $X_i \in \Gamma(\mathfrak{g})$ satisfying $Y_i \rightarrow X_i$ for $i = 1, 2$. Fix a base pair (Y_0, X_0) such that $Y_0 \rightarrow X_0$, provided such a pair exists. Choose $\Delta(Y_0, X_0) \in \mathbb{C}^\times$ arbitrarily. For every pair (Y, X) such that $Y \rightarrow X$, set

$$\Delta(Y, X) = \Delta(Y, X, Y_0, X_0) \Delta(Y_0, X_0).$$

This makes sense because, as we shall see, for such a pair one has

$$\Delta(Y, X, Y, X) = 1. \tag{7.1}$$

To define the relative transfer factor, we need a quasi-split inner form G^* of G , as in §2.5 and (2.1), together with a pinning of G^* , as in §4.6. We then have a section of χ_{G^*} by Theorem 4.2. Composing it with $\mathfrak{c}_H \rightarrow \mathfrak{c}_{G^*}$ from (4.1) and with χ_H gives a map

$$\Sigma_G(\mathfrak{h}) \longrightarrow X_- + \mathfrak{g}_{X_+}(F),$$

which we denote by $Y \mapsto Y^*$. We leave it to the reader to check that the relative transfer factor is independent of these auxiliary choices.

7.3. Underlying the datum (H, κ, ξ) are maximal subtori $T_G \subset G$ and $T_H \subset H$. Moreover, κ induces a character of $H^1(F, T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}})$; see §3.1. Set

$$U = (T_{G_{\text{sc}}} \times T_{G_{\text{sc}}}) / \{(z, z^{-1}) \mid z \in Z_{G_{\text{sc}}}\}.$$

Multiplication induces an evident morphism $U \rightarrow T_{G_{\text{sc}}}$ and, more generally, a morphism of complexes

$$[T_{H_{\text{sc}}} \times T_{H_{\text{sc}}} \rightarrow U] \longrightarrow [T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}}].$$

Thus we have a map

$$H^1(F, T_{H_{\text{sc}}} \times T_{H_{\text{sc}}} \rightarrow U) \longrightarrow H^1(F, T_{H_{\text{sc}}} \rightarrow T_{G_{\text{sc}}}).$$

Let κ_U be the character that κ induces on the first group.

We wish to define $\Delta(Y_1, X_1, Y_2, X_2)$ for a quadruple as above. By the definition of Y_i^* , there is $g_i \in G_{\text{sc}}^*$ such that $\text{Ad}(g_i)T_G = T_{Y_i^*}$ and the conditions of §4.4 hold. Set

$$U_{1,2} = (T_{Y_1^*, G_{\text{sc}}} \times T_{Y_2^*, G_{\text{sc}}}) / \{(z, z^{-1}) \mid z \in Z_{G_{\text{sc}}}\}.$$

By a variant of the argument following (3.3), the pair $(\text{Ad}(g_1^{-1}), \text{Ad}(g_2^{-1}))$ induces a Γ -equivariant isomorphism

$$\begin{aligned} X_*(T_{Y_1, H_{\text{sc}}} \times T_{Y_2, H_{\text{sc}}}) \backslash X_*(U_{1,2}) \\ \longrightarrow X_*(T_{H_{\text{sc}}} \times T_{H_{\text{sc}}}) \backslash X_*(U), \end{aligned}$$

and hence an isomorphism

$$\begin{aligned} H^1(F, T_{Y_1, H_{\text{sc}}} \times T_{Y_2, H_{\text{sc}}} \rightarrow U_{1,2}) \\ \longrightarrow H^1(F, T_{H_{\text{sc}}} \times T_{H_{\text{sc}}} \rightarrow U). \end{aligned}$$

Therefore κ_U induces a character on $H^1(F, T_{Y_1, H_{\text{sc}}} \times T_{Y_2, H_{\text{sc}}} \rightarrow U_{1,2})$, and hence a character $\kappa_{1,2}$ on $H^1(F, U_{1,2})$.

Since $Y_i \rightarrow X_i$ by assumption, we have $\chi_G(X_i) = \chi_{G^*}(Y_i^*)$. Thus there is $x_i \in G_{\text{sc}}^*$ such that

$$\text{Ad}(x_i) \circ \varphi(X_i) = Y_i^*.$$

For every $\sigma \in \Gamma$, let $u_\sigma \in G_{\text{sc}}^*$ satisfy $\varphi\sigma(\varphi)^{-1} = \text{Ad}(u_\sigma)$. Then

$$(x_1 u_\sigma \sigma(x_1)^{-1}, \sigma(x_2) u_\sigma^{-1} x_2^{-1})$$

defines a 1-cochain with values in $U_{1,2}$, whose class in $H^1(F, U_{1,2})$ is denoted by $\text{inv}(Y_1, X_1, Y_2, X_2)$. Set

$$\Delta(Y_1, X_1, Y_2, X_2) = \langle \kappa_{1,2}, \text{inv}(Y_1, X_1, Y_2, X_2) \rangle.$$

7.4. Let $X^* \in \mathfrak{g}_{\text{reg}}^*(F)$. For $i = 1, 2$, let $X_i \in \mathfrak{g}(F)$ be such that there exists $x_i \in G_{\text{sc}}^*$ with $\text{Ad}(x_i) \circ \varphi(X_i) = X^*$. Then the product

$$x_1 u_\sigma \sigma(x_1)^{-1} \sigma(x_2) u_\sigma^{-1} x_2^{-1}$$

defines a Galois 1-cochain with values in $T_{X^*, \text{sc}}$. Its class in $H^1(F, T_{X^*, \text{sc}})$ is denoted by $\text{inv}(X^*; X_1, X_2)$. One easily checks the following formula: for every $Y \in \mathfrak{h}_{G\text{-reg}}(F)$ such that $Y \rightarrow X^*$,

$$\Delta(Y, X_1) = \Delta(Y, X_2) \langle \kappa_Y, \text{inv}(X^*; X_1, X_2) \rangle, \quad (7.2)$$

where κ_Y was defined at the end of §4.4.

8. SMOOTH TRANSFER AND FOURIER TRANSFORMATION: THE p -ADIC CASE

8.1. Throughout this section, F is a p -adic field.

8.2. Two stably conjugate elements of $\mathfrak{g}_{\text{reg}}(F)$ have “stably conjugate,” hence isomorphic, centralizer tori. We require the Haar measures on these tori to correspond under this isomorphism.

8.3. Let $X \in \Sigma(\mathfrak{g})$. The *stable orbital integral* S_X is the invariant distribution defined by

$$S_X(f) = \sum_{\substack{Z \in \Gamma(\mathfrak{g}) \\ \chi(Z) = \chi(X)}} I_Z(f).$$

A distribution T is *stable* if it belongs to the weak closure of the space of stable orbital integrals, that is, if $S_X(f) = 0$ for every $X \in \Sigma(\mathfrak{g})$ implies $T(f) = 0$.

To emphasize that these objects are attached to G , we will write I^G , S^G , and so on.

8.4. Let (H, κ, ξ) be an endoscopic datum for G . For every $Y \in \Sigma_G(\mathfrak{h})$, the *endoscopic orbital integral* $I_Y^{G,H}$ is the invariant distribution defined by

$$I_Y^{G,H}(f) = \sum_{X \in \Gamma(\mathfrak{g})} \Delta(Y, X) I_X(f).$$

8.5. Here is Waldspurger’s smooth transfer theorem.

Theorem 8.1. *For every $f \in C_c^\infty(\mathfrak{g}(F))$, there exists $f^H \in C_c^\infty(\mathfrak{h}(F))$ such that, for every $Y \in \Sigma_G(\mathfrak{h})$,*

$$I_Y^{G,H}(f) = S_Y^H(f^H).$$

We say that f^H is a transfer of f .

One can show that, if T is a stable distribution on $\mathfrak{h}(F)$, then $T(f^H)$ depends only on f . This gives a distribution on $\mathfrak{g}(F)$, the *dual endoscopic transfer*.

8.6. Suppose that G is unramified: by definition, it is quasi-split and splits over an unramified extension. Bruhat–Tits theory shows that there is a smooth group scheme $\mathcal{G}$ over the ring of integers $\mathcal{O}_F$, with connected reductive fibers and generic fiber G . Let 1_G be the characteristic function of the lattice $\text{Lie}(\mathcal{G})(\mathcal{O}_F)$ in $\mathfrak{g}(F)$. Such a lattice is called *hyperspecial*. If H is unramified, we likewise have a function 1_H . The following theorem, due to Waldspurger for SL_n , to Laumon–Ngô for unitary groups, and to Ngô in general, holds for every choice of integral models as above: it is the Langlands–Shelstad fundamental lemma.

Theorem 8.2. *There is a constant $c \neq 0$ such that, for every $Y \in \Sigma_G(\mathfrak{h})$,*

$${}_c I_Y^{G,H}(1_G) = S_Y^H(1_H).$$

8.7. Harish-Chandra emphasized the analogy between harmonic analysis on the group G and on its Lie algebra $\mathfrak{g}$. On both sides, of course, there are orbital integrals attached to regular semisimple elements. There are, however, other distributions on the group, namely the traces of tempered representations. Their analogues on the Lie algebra are the Fourier transforms of orbital integrals.

Waldspurger carried this analogy further. On the group, tempered representations are organized into L -packets whose associated traces should be stable distributions. According to Waldspurger, the analogue of an L -packet is a stable conjugacy class. In particular, one expects the Fourier transform $\widehat{S}_X^G$, for $X \in \Sigma(\mathfrak{g})$, to be stable. Waldspurger proved the following theorem.

Theorem 8.3. *The Fourier transform of a stable distribution is stable.*

8.8. As the bilinear form on $\mathfrak{h}(F)$, as in §5.4, choose the form whose restriction to $\mathfrak{t}_H$ is $\langle \iota(\cdot), \iota(\cdot) \rangle$; see §3.1. Using the same character ψ gives a Fourier transformation. We can then consider the dual transfer of $\widehat{S}_Y^H$. What is it? Waldspurger's answer is the following.

Theorem 8.4. *There is a constant $c \neq 0$ such that, for every $Y \in \Sigma_G(\mathfrak{h})$ and every $f \in C_c^\infty(\mathfrak{g}(F))$,*

$$c \widehat{I}_Y^{G,H}(f) = \widehat{S}_Y^H(f^H),$$

where f^H is a transfer of f . In other words, transfer commutes with Fourier transformation up to a constant.

8.9. Theorem 8.4 is equivalent to the following theorem, also due to Waldspurger. The forward implication follows from the local constancy of the functions $\widehat{i}^G$ and from Lemma 6.1, which allows one to isolate orbits.

Theorem 8.5. *There is a constant $c \neq 0$ such that, for every $Y \in \Sigma_G(\mathfrak{h})$ and every $Z \in \Gamma(\mathfrak{g})$, the expression*

$$\sum_{X \in \Gamma(\mathfrak{g})} \Delta(Y, X) \widehat{i}^G(X, Z)$$

is equal to

$$c \sum_{Z' \in \Gamma(\mathfrak{h})} \sum_{Y'} \frac{\widehat{i}^H(Y', Z') \Delta(Z', Z)}{|\ker(H^1(F, T_{Z'}) \rightarrow H^1(F, H))|}, \quad (8.1)$$

where the sum over Y' runs over the classes $Y' \in \Gamma(\mathfrak{h})$ stably conjugate to Y .

8.10. Remarkably, Theorem 8.5 implies the existence of transfer in Theorem 8.1. Let us sketch why. Fix a compact subset ω of $\mathfrak{h}$. For every compact $K \subset \mathfrak{g}(F)$, Howe's finiteness theorem, Theorem 5.1, implies that the space $V_{\omega, K}$ spanned by the functions

$$Y \in \omega \cap \mathfrak{h}_{G\text{-reg}}(F) \mapsto \sum_{X \in \Gamma(\mathfrak{g})} \Delta(Y, X) \widehat{i}^G(X, Z), \quad Z \in K \cap \mathfrak{g}_{\text{reg}}(F),$$

is finite-dimensional. By Theorem 8.5, this space is contained in the span of the functions in (8.1), and the latter is contained in the space of functions

$$Y \in \omega \cap \mathfrak{h}_{G\text{-reg}}(F) \mapsto S_Y^H(f'), \quad f' \in C_c^\infty(\mathfrak{h}(F));$$

to see this, use Lemma 6.1.

Take $f \in C_c^\infty(\mathfrak{g}(F))$. The integration formula in Theorem 6.3 implies that the function

$$Y \in \omega \cap \mathfrak{h}_{G\text{-reg}}(F) \longmapsto I_Y^{G,H}(f)$$

belongs to $V_{\omega,K}$ for a suitable K . But, as we have said, every element of $V_{\omega,K}$ has the form $S_Y^H(f')$ for some $f' \in C_c^\infty(\mathfrak{h}(F))$.

This gives local transfer near every point. A partition of unity allows us to glue the resulting transfers.

8.11. The use of an exponential map and descent to centralizers shows that transfer for groups follows from transfer for the Lie algebras of the centralizers of semisimple elements of the group. This requires checking various descent formulas for transfer factors.

9. ORBITAL INTEGRALS ON COMPLEX LIE ALGEBRAS

9.1. In this section, the field is $\mathbb{C}$. Over this field, stable conjugacy is the same as conjugacy. Let (H, κ, ξ) be an endoscopic datum for G . The transfer factor is either a fixed nonzero constant or zero.

9.2. Let $B \subset G$ be a Borel subgroup, $T \subset B$ a maximal subtorus, and $K \subset G(\mathbb{C})$ a maximal compact subgroup in “good position” with respect to T . Let U_B be the unipotent radical of B . Let $\mathcal{S}(\mathfrak{g}(\mathbb{C}))$ be the Schwartz space of $\mathfrak{g}(\mathbb{C})$. For $X \in \Gamma(\mathfrak{g})$ and $f \in \mathcal{S}(\mathfrak{g}(\mathbb{C}))$, define the orbital integral $I_X(f)$ as in §6.3. This integral converges. In the complex case, the Iwasawa decomposition expresses an orbital integral as a “spherical” constant term. More precisely, with choices of Haar measures that we do not specify here, for every $X \in \mathfrak{t}_{\text{reg}}(\mathbb{C})$ one has

$$I_X(f) = f_B(X),$$

where the “constant term” is defined, for every $X \in \mathfrak{t}(\mathbb{C})$, by

$$f_B(X) = \int_{K \times u_B(\mathbb{C})} f(\text{Ad}(k)(X + N)) dN dk.$$

We have the following theorem, a special case of a theorem of Bouaziz; see [1].

Theorem 9.1. *Let $W = W^G(T)$ be the Weyl group. The map*

$$f \in \mathcal{S}(\mathfrak{g}(\mathbb{C})) \longmapsto f_B \in \mathcal{S}(\mathfrak{t}(\mathbb{C}))^W$$

is surjective.

9.3. Fix a character ψ and a form as in §5.4, giving a Fourier transformation on $\mathcal{S}(\mathfrak{g}(\mathbb{C}))$. We require the forms on $\mathfrak{g}$ and $\mathfrak{h}$ to be compatible, as in §8.8: they agree on the Lie algebra of “the” common torus, where we say “the” because all maximal tori are conjugate. The following result is then easily deduced from the preceding theorem.

Theorem 9.2. *Let $f \in \mathcal{S}(\mathfrak{g}(\mathbb{C}))$. Then*

– *there is a transfer $f^H \in \mathcal{S}(\mathfrak{h}(\mathbb{C}))$ such that, for every $Y \in \Gamma_G(\mathfrak{h})$,*

$$I_Y(f^H) = I_Y^{G,H}(f);$$

– *transfer commutes with Fourier transformation.*

10. PRE-STABILIZATION OF THE TRACE FORMULA FOR LIE ALGEBRAS

10.1. In this section, unless otherwise stated, F is a number field.

10.2. An analogue of the elliptic part of the Arthur–Selberg trace formula. Let $\mathcal{S} = \mathcal{S}(\mathfrak{g}(\mathbb{A}))$ be the Schwartz–Bruhat space of $\mathfrak{g}(\mathbb{A})$: it is the space spanned by functions $f_\infty \otimes f^\infty$, where f_∞ is a Schwartz function on $\mathfrak{g}(F \otimes \mathbb{R})$ and f^∞ is the characteristic function of a compact open subgroup of $\mathfrak{g}(\mathbb{A}_f)$; here $\mathbb{A}_f$ denotes the finite adeles. Write $G(\mathbb{A})^1$ for the subgroup of $G(\mathbb{A})$ fitting into the exact sequence

$$1 \longrightarrow G(\mathbb{A})^1 \longrightarrow G(\mathbb{A}) \longrightarrow \mathrm{Hom}(X^*(G), \mathbb{R}_+^\times) \longrightarrow 1,$$

where $X^*(G)$ is the group of rational characters of G defined over F , and the nonobvious map sends $g \in G(\mathbb{A})$ to the homomorphism $\chi \mapsto |\chi(g)|_\mathbb{A}$.

Equip $G(\mathbb{A})$ with Tamagawa measure and $\mathrm{Hom}(X^*(G), \mathbb{R}_+^\times)$ with the measure induced by the usual measure on $\mathbb{R}_+^\times$. This gives a measure on $G(\mathbb{A})^1$. The quotient $G(F) \backslash G(\mathbb{A})^1$ has finite volume, and the *Tamagawa number* of G is

$$\tau(G) = \mathrm{vol}(G(F) \backslash G(\mathbb{A})^1).$$

The calculation of Tamagawa numbers was completed with Kottwitz’s proof of Weil’s conjecture. We state the result in the following form; see [5, Corollary 1.7.4, p. 39]:

$$\tau(G) = \frac{|H_{\mathrm{ab}}^1(\mathbb{A}/F, G)|}{|\mathrm{Ker}_{\mathrm{ab}}^1(F, G)|}.$$

This expression also applies to subtori of G .

Theorem 10.1. *For every $f \in \mathcal{S}$, the integral*

$$\int_{G(F) \backslash G(\mathbb{A})^1} \sum_{X \in \mathfrak{g}_{\mathrm{ell}}(F)} |f(x^{-1}Xx)| \, dx < \infty$$

converges.

For the proof, we refer to [9, §10.8]. It rests on reduction theory, which provides a substitute for a fundamental domain for $G(F) \backslash G(\mathbb{A})^1$.

We may therefore set

$$J^G(f) = \int_{G(F) \backslash G(\mathbb{A})^1} \sum_{X \in \mathfrak{g}_{\mathrm{ell}}(F)} f(\mathrm{Ad}(x^{-1})X) \, dx.$$

We also write

$$J^G(f) = \sum_{X \in \Gamma_{\mathrm{ell}}(\mathfrak{g})} \tau(T_X) I_X^G(f),$$

where the global orbital integral is defined by

$$I_X^G(f) = \int_{T_X(\mathbb{A}) \backslash G(\mathbb{A})} f(\mathrm{Ad}(x^{-1})X) \, dx. \tag{10.1}$$

10.3. In the preceding paragraph, we expressed $J^G(f)$ in terms of rational conjugacy classes. We would like an expression in terms of local stable conjugacy classes. This will be possible, but with an error term. We must first introduce the Langlands obstruction.

10.4. The Langlands obstruction. We seek a cohomological condition ensuring that an element of $\mathfrak{g}^*(F)$ which transfers to $\mathfrak{g}$ everywhere locally also transfers globally.

More precisely, let $X^* \in \mathfrak{g}^*(F)$ be regular semisimple, and suppose that there exist $x \in G^*(\bar{\mathbb{A}})$ and $X \in \mathfrak{g}(\mathbb{A})$ such that

$$\mathrm{Ad}(x) \circ \varphi(X) = X^*.$$

We may, and do, assume that $x \in G_{\mathrm{sc}}^*(\bar{\mathbb{A}})$; see [9, §10.2]. Let u_σ be a cochain with values in G_{sc}^* such that $\varphi\sigma(\varphi)^{-1} = \mathrm{Ad}(u_\sigma)$ for every $\sigma \in \Gamma$. Let T be the centralizer of X^* in G^* . The cochain $xu_\sigma\sigma(x)^{-1} \in T_{\mathrm{sc}}(\bar{\mathbb{A}})$ defines a 1-cocycle class in $H^1(\mathbb{A}_F/F, T_{\mathrm{sc}})$, denoted by $\widetilde{\mathrm{inv}}(X^*, X)$. Since x is well defined up to left translation by an element of $T_{\mathrm{sc}}(\bar{\mathbb{A}})$, the class $\widetilde{\mathrm{inv}}(X^*, X)$ depends only on X^* and X . Write $\mathrm{inv}(X^*, X)$ for its image in $\mathcal{E}(T, G, \mathbb{A}/F)$.

The following theorem is due to Langlands.

Theorem 10.2. *There exists $g \in G(\mathbb{A})$ such that $\mathrm{Ad}(g)X \in \mathfrak{g}(F)$ if and only if $\mathrm{inv}(X^*, X) = 1$.*

Proof. The condition is necessary. Indeed, if $g \in G(\mathbb{A})$ satisfies $\mathrm{Ad}(g)X \in \mathfrak{g}(F)$, we are immediately reduced to the case $X \in \mathfrak{g}(F)$. The set of $y \in G_{\mathrm{sc}}^*$ such that

$$\mathrm{Ad}(y) \circ \varphi(X) = X^*$$

is then a nonempty subvariety of G_{sc}^* , since it has an $\bar{\mathbb{A}}$ -point. Thus we may take $x \in G_{\mathrm{sc}}^*(\bar{F})$. We see that $\widetilde{\mathrm{inv}}(X^*, X) = 1$, proving the first implication.

The converse is more involved. Since $\mathrm{inv}(X^*, X) = 1$, there exist a 1-cocycle β_σ with values in $T_{\mathrm{sc}}(\bar{\mathbb{A}})$, an element $y \in T(\bar{\mathbb{A}})$, and a 1-cochain t_σ with values in $T_{\mathrm{sc}}(\bar{F})$ such that

$$\beta_\sigma = y^{-1}\sigma(y)$$

—here we do not distinguish β_σ from its image in $T(\bar{\mathbb{A}})$ —and

$$xu_\sigma\sigma(x)^{-1} = \beta_\sigma t_\sigma$$

for every $\sigma \in \Gamma$. Since the coboundary $\partial\beta$ is trivial, $\partial u = \partial t$. It follows that the cochain $t_\sigma u_\sigma^{-1}$ is a cocycle “with values in G_{sc} ,” a somewhat abusive expression meaning that $\varphi^{-1}(t_\sigma u_\sigma^{-1})$ is a cocycle with values in G_{sc} .

The cochain $yxu_\sigma\sigma(yx)^{-1}u_\sigma^{-1}$ defines a locally trivial 1-cocycle with values in G ; its class therefore belongs to $\mathrm{Ker}^1(F, G)$. Its image in the abelianization $\mathrm{Ker}_{\mathrm{ab}}^1(F, G)$ is trivial: it is the class of the hypercochain $(1, t_\sigma u_\sigma^{-1})$, which is cohomologous to the trivial class. But the abelianization map

$$\mathrm{Ker}^1(F, G) \longrightarrow \mathrm{Ker}_{\mathrm{ab}}^1(F, G)$$

is an isomorphism. Hence the cochain $yxu_\sigma\sigma(yx)^{-1}u_\sigma^{-1}$ is trivial: there is $a \in G^*$ such that

$$yxu_\sigma\sigma(yx)^{-1}u_\sigma^{-1} = au_\sigma\sigma(a)^{-1}u_\sigma^{-1}.$$

Let $Y \in \mathfrak{g}(\bar{F})$ satisfy $\mathrm{Ad}(a) \circ \varphi(Y) = X^*$. Then $Y = \mathrm{Ad}(b)X$, where $b = \varphi^{-1}(a^{-1}yx)$, and one checks that $b \in G(\mathbb{A})$. Consequently,

$$Y \in \mathfrak{g}(\bar{F}) \cap \mathfrak{g}(\mathbb{A}) = \mathfrak{g}(F),$$

as required. □

10.5. Pre-stabilization. To express $J^G(f)$ in terms of local stable conjugacy classes, we use Fourier inversion on the group $\mathcal{E}(T_{X^*}, G^*, \mathbb{A}/F)$.

We identify $\Gamma_{\text{ell}}(\mathfrak{g})$ and $\Sigma_{\text{ell}}(\mathfrak{g}^*)$ with sets of representatives. For every $X^* \in \Sigma_{\text{ell}}(\mathfrak{g}^*)$, set

$$\Gamma_{X^*}(\mathfrak{g}) = \{X \in \Gamma_{\text{ell}}(\mathfrak{g}) \mid \exists g \in G_{\text{sc}}^*, \text{Ad}(g) \circ \varphi(X) = X^*\}.$$

This set may be empty. If it is not, its elements represent the ordinary conjugacy classes in a single stable class of $\mathfrak{g}(F)$ determined by X^* . We have the disjoint union

$$\Gamma_{\text{ell}}(\mathfrak{g}) = \coprod_{X^* \in \Sigma_{\text{ell}}(\mathfrak{g}^*)} \Gamma_{X^*}(\mathfrak{g}).$$

Introduce also

$$\begin{aligned} \Gamma_{X^*}(\mathfrak{g}(\mathbb{A})) &= \{X \in \mathfrak{g}(\mathbb{A}) \mid \exists g \in G_{\text{sc}}^*(\overline{\mathbb{A}}), \\ &\quad \text{Ad}(g) \circ \varphi(X) = X^*\}/G(\mathbb{A}). \end{aligned}$$

Theorem 10.2 implies that the evident map

$$\Gamma_{X^*}(\mathfrak{g}) \longrightarrow \{X \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A})) \mid \text{inv}(X^*, X) = 1\} \quad (10.2)$$

is surjective. It is not injective. Let $X \in \Gamma_{X^*}(\mathfrak{g})$. Then $\Gamma_{X^*}(\mathfrak{g})$ is naturally in bijection with $\mathcal{D}(T_X, G, F)$. An element of $\Gamma_{X^*}(\mathfrak{g})$ maps to the class of X under (10.2) if and only if it corresponds to an element of

$$\begin{aligned} &\text{Ker}\left(\mathcal{D}(T_X, G, F) \longrightarrow \mathcal{D}(T_X, G, \mathbb{A})\right) \\ &= \text{Ker}\left(\text{Ker}^1(F, T_X) \longrightarrow \text{Ker}^1(F, G)\right). \end{aligned}$$

Using the natural bijection between $\text{Ker}^1(F, G)$ and $\text{Ker}_{\text{ab}}^1(F, G)$, and the fact that abelianized cohomology is unaffected by inner twisting, we see that all fibers of (10.2) have the same cardinality, namely

$$d(X^*) = \left| \text{Ker}\left(\text{Ker}^1(F, T_{X^*}) \longrightarrow \text{Ker}^1(F, G^*)\right) \right|.$$

The expression (10.1) defining $I_X^G(f)$ also applies to every $X \in \mathfrak{g}(\mathbb{A})$. As a function of X , the orbital integral is $G(\mathbb{A})$ -invariant. Finally, if $X \in \Gamma_{X^*}(\mathfrak{g})$, one checks that $\tau(T_X) = \tau(T_{X^*})$.

We may therefore write

$$\begin{aligned} J^G(f) &= \sum_{X^* \in \Sigma_{\text{ell}}(\mathfrak{g}^*)} \tau(T_{X^*}) d(X^*) \sum_{\substack{X \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A})) \\ \text{inv}(X^*, X) = 1}} I_X^G(f) \\ &= \sum_{X^* \in \Sigma_{\text{ell}}(\mathfrak{g}^*)} \frac{\tau(T_{X^*}) d(X^*)}{|\mathcal{E}(T_{X^*}, G^*, \mathbb{A}/F)|} \\ &\quad \cdot \sum_{X \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A}))} \sum_{\kappa \in \mathcal{K}(T_{X^*}, G^*, F)} \langle \kappa, \text{inv}(X^*, X) \rangle I_X^G(f). \end{aligned}$$

We now use the following exact sequence from [5, Proposition 1.8.4, p. 42]:

$$\begin{aligned} 1 &\longrightarrow \mathrm{Ker}\left(\mathrm{Ker}^1(F, T_{X^*}) \rightarrow \mathrm{Ker}_{\mathrm{ab}}^1(F, G^*)\right) \\ &\longrightarrow \mathrm{Ker}^1(F, T_{X^*}) \longrightarrow \mathrm{Ker}_{\mathrm{ab}}^1(F, G^*) \\ &\longrightarrow \mathcal{E}(T_{X^*}, G^*, \mathbb{A}/F) \longrightarrow H^1(\mathbb{A}/F, T_{X^*}) \\ &\longrightarrow H_{\mathrm{ab}}^1(\mathbb{A}/F, G^*) \longrightarrow H^2(\mathbb{A}/F, T_{X^*, \mathrm{sc}}) \longrightarrow 1. \end{aligned}$$

Since $T_{X^*, \mathrm{sc}}$ is elliptic, the Tate–Nakayama isomorphism shows that $H^2(\mathbb{A}/F, T_{X^*, \mathrm{sc}}) = 1$. Using the natural bijection between $\mathrm{Ker}^1(F, G^*)$ and its abelianization, we obtain

$$\frac{\tau(T_{X^*})d(X^*)}{|\mathcal{E}(T_{X^*}, G^*, \mathbb{A}/F)|} = \tau(G^*).$$

The set $\Gamma_{X^*}(\mathfrak{g}(\mathbb{A}))$ is infinite. There is a finite set S of places such that G is defined over $\mathfrak{o}_S$, the ring of elements of F integral outside S , and such that, for all $X, X' \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A}))$ with $I_X^G(f) \neq 0$, the elements X_v and X'_v are conjugate under $G(\mathfrak{o}_v)$ for $v \notin S$; see [9, Lemma 7.2]. It follows that

$$\{X \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A})) \mid I_X^G(f) \neq 0\}$$

is finite, so the order of summation over X and κ may be interchanged. We find

$$J^G(f) = \tau(G^*) \sum_{X^* \in \Sigma_{\mathrm{ell}}(\mathfrak{g}^*)} \sum_{\kappa \in \mathcal{K}(T_{X^*}, G^*, F)} I_{X^*}^{G, \kappa}(f), \quad (10.3)$$

where

$$I_{X^*}^{G, \kappa}(f) = \sum_{X \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A}))} \langle \kappa, \mathrm{inv}(X^*, X) \rangle I_X^G(f).$$

10.6. In this paragraph, F is local or global.

Start with an endoscopic triple (H, κ, ξ) and an element $Y \in \mathfrak{h}_{G\text{-reg}}(F)$. There exists $X^* \in \Sigma(\mathfrak{g}^*)$ such that $Y \rightarrow X^*$. In §4.4, we explained how to define an element $\kappa_Y \in \mathcal{K}(T_{X^*}, G^*, F)$. We state the following lemma.

Lemma 10.3. *For every $X^* \in \Sigma_{\mathrm{ell}}(\mathfrak{g}^*)$ and $\kappa \in \mathcal{K}(T_{X^*}, G^*, F)$, there exist an elliptic endoscopic triple (H, κ, ξ) and $Y \in \Sigma(\mathfrak{h})$ such that $Y \rightarrow X^*$ and $\kappa_Y = \kappa$.*

Any two such data (H, κ, ξ, Y) and (H', κ', ξ', Y') are equivalent as endoscopic data for G by an isomorphism carrying Y to Y' . Such an isomorphism is unique up to an inner automorphism.

Proof. We discuss only existence; a complete proof is given in [3, Lemma 9.7]. Let $X^* \in \mathfrak{g}_{\mathrm{ell}}^*(F)$, and set $T = T_{X^*}$. Let $\kappa \in \mathcal{K}(T_{X^*}, G^*, F)$. In particular, κ defines a character of $H^1(C_F, T_{\mathrm{sc}})$. Fix a finite Galois extension E of F splitting T . Tate–Nakayama duality identifies $H^1(C_F, T_{\mathrm{sc}})$ with $\widehat{H}^{-1}(E/F, X_*(T_{\mathrm{sc}}))$. Since $X_*(T_{\mathrm{sc}})^\Gamma = 1$ by ellipticity, we also have

$$H^1(C_F, T_{\mathrm{sc}}) \simeq H_0(E/F, X_*(T_{\mathrm{sc}})).$$

Thus we may regard κ as a character of $H_0(E/F, X_*(T_{\mathrm{sc}}))$. The root datum

$$(X^*(T_G), R_\kappa, X_*(T_G), R_\kappa^\vee)$$

is stable under Γ . Moreover, the action of Γ carries a base to a base. This root datum is therefore the root datum of a group H . One readily completes these data to an endoscopic

triple (H, κ, ξ) . The torus T identifies with a subtorus of H , and this identification carries X^* to a G -regular element $Y \in \mathfrak{h}(F)$. This constructs the quadruple (H, κ, ξ, Y) . $\square$

10.7. Normalization of transfer factors. Let F be a global field and (H, κ, ξ) an endoscopic datum for G . Let V be the set of places of F . For every $v \in V$, we obtain in the evident way a local endoscopic datum (H_v, κ, ξ) for G_v , where G_v denotes the group over the completion F_v obtained by extending scalars.

Let $Y_i \in \mathfrak{h}_{G\text{-reg}}(F)$ and $X_i^* \in \mathfrak{g}_{\text{reg}}^*(F)$ satisfy $Y_i \rightarrow X_i^*$ for $i = 1, 2$. Suppose that $\Gamma_{X_i^*}(\mathfrak{g}(\mathbb{A})) \neq \emptyset$, and choose $X_i \in \Gamma_{X_i^*}(\mathfrak{g}(\mathbb{A}))$.

At every place v , there is a relative transfer factor denoted by Δ_{G_v, H_v} , or simply Δ_v ; see Section 7. One verifies the following formula; see [7, Section 6]:

$$\begin{aligned} & \langle \kappa_{Y_1}, \text{inv}(X_1^*, X_1) \rangle \langle \kappa_{Y_2}, \text{inv}(X_2^*, X_2) \rangle^{-1} \\ &= \prod_{v \in V} \Delta_v(Y_1, X_1, Y_2, X_2), \end{aligned}$$

where $\Delta_v(Y_1, X_1, Y_2, X_2) = 1$ for almost every v .

Normalize the transfer factor by taking $(Y_2, X_{2,v})$ as base pair. We require

- $\Delta_v(Y_2, X_{2,v}) = 1$ for almost every v ;
- $\prod_{v \in V} \Delta_v(Y_2, X_2) = \langle \kappa_{Y_2}, \text{inv}(X_2^*, X_2) \rangle$.

Once the transfer factors are normalized in this way, these last two formulas hold for all $Y \in \mathfrak{h}_{G\text{-reg}}(F)$, $X^* \in \mathfrak{g}_{\text{reg}}^*(F)$, and $X \in \Gamma_{X^*}(\mathfrak{g}(\mathbb{A}))$ such that $Y \rightarrow X^*$.

10.8. Return to the notation at the end of §10.5. Let $X^* \in \Sigma_{\text{ell}}(\mathfrak{g}^*)$ and $\kappa \in \mathcal{K}(T_{X^*}, G^*, F)$. Let (H, κ, ξ) be an endoscopic datum and $Y \in \mathfrak{h}_{G\text{-reg}}(F)$ such that $Y \rightarrow X^*$ and $\kappa_Y = \kappa$, as in Lemma 10.3. Suppose that $f = \bigotimes_{v \in V} f_v$. Using the preceding normalization of transfer factors, we find

$$I_{X^*}^{G, \kappa}(f) = \prod_{v \in V} I_Y^{G_v, H_v}(f_v),$$

where $I_Y^{G_v, H_v}(f_v) = 1$ for almost every $v \in V$; see [9, Section 7 and §10.2]. At this point one must establish compatibility relations between local and global Haar measures. We will not discuss them.

11. WALDSPURGER'S RESULT

11.1. Let F be a global field and u a finite place of F . Let (H, κ, ξ) be an endoscopic datum. Assume the following conditions.

1. F is totally imaginary.
2. For every $v \in V$, there exist $Y \in \mathfrak{h}_{G\text{-reg}}(F_v)$ and $X \in \mathfrak{g}_{\text{reg}}(F_v)$ such that $Y \rightarrow X$.
3. The place u is inert in $F[H]$, the smallest extension over which H splits.
4. The datum (H_u, κ, ξ) is elliptic.

Here is the theorem proved by Waldspurger in [9].

Theorem 11.1. *Let $w \in V$ be a finite place distinct from u . Then Theorem 8.4 holds for the pair (G_w, H_w) .*

From this, globalization or descent arguments show that Theorem 8.4 holds in full generality; see [9, Sections 4, 5, 6, and 11]. Note that proving Theorem 11.1 requires the fundamental lemma, Theorem 8.2, for almost every pair (G_v, H_v) .

11.2. Fix a finite place $w \neq u$ of F . Let

$$\langle \cdot, \cdot \rangle : \mathfrak{g}(F) \times \mathfrak{g}(F) \longrightarrow F$$

be a nondegenerate symmetric bilinear form invariant under $G(F)$, and let $\psi : \mathbb{A}_F/F \rightarrow \mathbb{C}^\times$ be a continuous nontrivial character. As in §8.8, this gives an analogous form on $\mathfrak{h}(F)$.

Fix two $\mathcal{O}_F$ -lattices $\mathfrak{a} \subset \mathfrak{g}(F)$ and $\mathfrak{b} \subset \mathfrak{h}(F)$. They give $\mathcal{O}_{F_v}$ -lattices $\mathfrak{a}_v \subset \mathfrak{g}(F_v)$ and $\mathfrak{b}_v \subset \mathfrak{h}(F_v)$. There is a finite set S_1 of places containing the archimedean places and the places u, w such that, for every $v \notin S_1$, the lattices $\mathfrak{a}_v$ and $\mathfrak{b}_v$ are hyperspecial and self-dual for the bicharacters $\psi(\langle \cdot, \cdot \rangle)$.

11.3. The set of equivalence classes of endoscopic data for the quasi-split inner form G^* of G that are unramified outside S_1 is finite; see [6, Lemma 8.12]. Fix a set $\mathcal{H}$ of representatives. Since H is unramified outside S_1 , we may, and do, assume that $(H, \kappa, \xi) \in \mathcal{H}$. Let $\mathcal{H}' \subset \mathcal{H}$ be the subset consisting of data (H', κ', ξ') such that $F[H'] \not\subset F[H]$, that is, such that H' is not split over $F[H]$. Choose

$$v = v_{(H', \kappa', \xi')} \in V \setminus S_1$$

such that its Frobenius in $\text{Gal}(F[H]F[H']/F)$ fixes $F[H]$ but not $F[H']$. Then H_v is split, whereas H'_v is not. Set

$$S_2 = \{v_{(H', \kappa', \xi')} \mid (H', \kappa', \xi') \in \mathcal{H}'\}, \quad S = S_1 \cup S_2.$$

11.4. We wish to prove that there is a constant $c \neq 0$ with the following property. For every $f_w \in C_c^\infty(\mathfrak{g}(F_w))$, every transfer $f_w^H \in C_c^\infty(\mathfrak{h}(F_w))$ of f_w , in the sense of Theorem 8.1, and every $Y_w \in \Sigma_G(\mathfrak{h})$, one has

$$c \widehat{I}_{Y_w}^{G_w, H_w}(f_w) = \widehat{S}_{Y_w}^{H_w}(f_w^H). \quad (11.1)$$

In fact, we shall see that

$$c = \prod_{\substack{v \in S \setminus V_\infty \\ v \neq w}} c_v^{-1},$$

where c_v is the constant introduced in condition 7 of Proposition 11.2.

Let $Y_w \in \mathfrak{h}_{G\text{-reg}}(F_w)$. Fix a compact open subset $\Omega_w \subset \mathfrak{h}_{G\text{-reg}}(F_w)$ containing Y_w on which both sides of the desired identity are constant.

11.5. Let V_∞ be the set of archimedean places. For every $v \in S \setminus V_\infty$ with $v \neq w$, fix a compact open subset $\Omega_v \subset \mathfrak{h}_{G\text{-reg}}(F_v)$ such that

1. every point of Ω_v is an image of an element of $\mathfrak{g}_{\text{reg}}(F_v)$; this is possible by condition 2 of §11.1;
2. if $v \in S_2$, every point of Ω_v has a split centralizer torus; this is possible because H_v is split at such a place;
3. every point of Ω_u is elliptic; this is compatible with condition 1 by condition 4 of §11.1 and Lemma 4.6.

11.6. For every maximal subtorus T of G^* defined over F , there is a natural map

$$\mathcal{E}(T, G^*, F_v) \longrightarrow \mathcal{E}(T, G^*, \mathbb{A}_F/F),$$

and hence a dual map

$$\mathcal{K}(T, G^*, \mathbb{A}_F/F) \longrightarrow \mathcal{K}(T, G^*, F_v),$$

denoted by $\kappa \mapsto \kappa_v$.

11.7. Choose $Y_0 \in \mathfrak{h}(F)$ such that

1. $Y_0 \in \Omega_v$ for every $v \in S \setminus V_\infty$;
2. $Y_0 \in \mathfrak{b}_v$ for every $v \in V \setminus S$.

Strong approximation guarantees the existence of Y_0 . Let $X_0^* \in \mathfrak{g}_{\text{reg}}^*(F)$ satisfy $Y_0 \rightarrow X_0^*$. One checks that X_0^* is regular semisimple and elliptic. Let $\mathcal{Y} \subset \Sigma_G(\mathfrak{h})$ be the finite subset of classes Y such that $Y \rightarrow X_0^*$, identified with a set of representatives. Of course, we assume that $Y_0 \in \mathcal{Y}$. At the end of §4.4, we explained how every element of $\mathcal{Y}$ defines an element $\kappa_Y \in \mathcal{K}(T_{X_0^*}, G^*, F)$. Set

$$\kappa_0 = \kappa_{Y_0}, \quad \mathcal{Y}_0 = \{Y \in \mathcal{Y} \mid \kappa_Y = \kappa_0\}.$$

11.8. Choice of functions outside S . For every $v \notin S$, let f_v , respectively f_v^H , be the characteristic function of $\mathfrak{a}_v$, respectively $\mathfrak{b}_v$. Let $c_v \neq 0$ be the constant supplied by Theorem 8.2.

11.9. Construction of functions at finite places. Let $v \in S$ be a finite place distinct from w . Let $X^* \in \mathfrak{g}_{\text{reg}}^*(F_v)$. Set

$$\mathfrak{g}(X^*, F_v) = \chi_G^{-1}(\chi_{G^*}(X^*)) \cap \mathfrak{g}(F_v).$$

Let $\kappa \in \mathcal{K}(T_{X^*}, G^*, F_v)$. A κ -function above X^* is a function $\theta : \mathfrak{g}(X^*, F_v) \rightarrow \mathbb{C}$ such that

- θ is $G(F_v)$ -invariant;
- for $X_1, X_2 \in \mathfrak{g}(X^*, F_v)$,

$$\theta(X_1) = \theta(X_2) \langle \kappa, \text{inv}(X^*; X_1, X_2) \rangle.$$

For such a function θ and $f \in C_c^\infty(\mathfrak{g}(F_v))$, set

$$I_{X^*}^\theta(f) = \sum_X \theta(X) I_X(f),$$

where the sum is over $X \in \mathfrak{g}(X^*, F_v)$ modulo conjugacy. When $\theta = \Delta(Y, \cdot)$, as in (7.2) of §7.4, for $Y \in \mathfrak{h}(F_v)$ such that $Y \rightarrow X^*$, we have

$$I_{X^*}^\theta(f) = I_Y^{G,H}(f).$$

When $G = G^*$ and φ is the identity, we can take

$$\theta = \langle \kappa, \text{inv}(X^*; \cdot, X^*) \rangle,$$

and then write $I_{X^*}^\kappa(f) = I_{X^*}^\theta(f)$.

We now come to the following construction of Waldspurger; see [9, Proposition 8.2]. Notation not recalled here is as in the preceding paragraphs.

Proposition 11.2. *There exist functions $f \in C_c^\infty(\mathfrak{g}_{\text{reg}}(F_v))$ and $f^H \in C_c^\infty(\mathfrak{h}_{G\text{-reg}}(F_v))$ with the following properties.*

1. If $X \in \text{supp}(f)$, respectively $Y \in \text{supp}(f^H)$, there exists $X' \in \mathfrak{t}_{X_0^*}(F_v)$ such that $\chi_{G^*}(X') = \chi_G(X)$, respectively such that $Y \rightarrow X'$.
2. f^H is a transfer of f .
3. Let $Z \in \mathfrak{g}_{\text{reg}}^*(F_v)$ and $\tau \in \mathcal{K}(T_Z, G^*, F_v)$. Suppose that there is no $g \in G^*$ such that conjugation by g induces an F_v -isomorphism from $(T_{X_0^*}, \kappa_0)$ to (T_Z, τ) . Then, for every τ -function θ above Z ,

$$I_Z^\theta(f) = 0.$$

4. Let $Y \in \mathfrak{h}_{G^*-\text{reg}}(F_v)$ and $\tau \in \mathcal{K}(T_Y, H, F_v)$. If $\tau \neq 1$, then

$$I_Y^\tau(f^H) = 0.$$

5. Let $\tau \in \mathcal{K}(T_{X_0^*}, G^*, F_v)$ and let θ be a nonzero τ -function above X_0^* . For $\tau \neq \kappa_0$, one has

$$I_{X_0^*}^\tau(\widehat{f}) = 0.$$

Moreover,

$$I_{X_0^*}^{\kappa_0}(\widehat{f}) \neq 0.$$

6. Let $Y \in \mathcal{Y}$ and $\tau \in \mathcal{K}(T_Y, H, F_v)$. Then

$$I_Y^\tau(\widehat{f^H}) = 0$$

whenever $Y \notin \mathcal{Y}_0$ or $\tau \neq 1$. Moreover,

$$I_Y^{G_v, H_v}(\widehat{f}) = 0$$

if $Y \notin \mathcal{Y}_0$.

7. There is a constant $c_v \neq 0$, depending only on $(G_v, H_v, \psi_v, \langle, \rangle)$, such that, for every $Y \in \mathcal{Y}_0$,

$$S_Y^{H_v}(\widehat{f^H}) = c_v I_Y^{G_v, H_v}(\widehat{f}) \neq 0.$$

We will not give the elaborate construction of these functions here. The reader can check that it is easy to construct pairs satisfying conditions 1–4. Conditions 5–7 are much harder to arrange. Their construction rests on an explicit expression “at infinity” for the functions $\widehat{i}^G$ of the theorem. More precisely, let $X, Z \in \mathfrak{g}_{\text{reg}}(F_v)$. Waldspurger shows that there is a nonnegative integer N such that, for every $\lambda \in F_v^\times$ with $v(\lambda) < -N$, the quantity $\widehat{i}^G(\lambda X, Z)$ has an explicit expression; see [8, Proposition VIII.1].

11.10. For every finite place $v \in S$ distinct from w , choose $f_v \in C_c^\infty(\mathfrak{g}_{\text{reg}}(F_v))$ and $f_v^H \in C_c^\infty(\mathfrak{h}_{G-\text{reg}}(F_v))$ satisfying the conclusions of Proposition 11.2.

11.11. Let $v \in V_\infty$ be an archimedean place. Recall that we assume $F_v \simeq \mathbb{C}$. Let $f_v \in \mathcal{S}(\mathfrak{g}(F_v))$ be arbitrary, and let $f_v^H \in \mathcal{S}(\mathfrak{h}(F_v))$ be a transfer of f_v , as in Theorem 9.2.

11.12. Set $f = \bigotimes_{v \in V} f_v$ and $f^H = \bigotimes_{v \in V} f_v^H$. Let $\text{Out}_G(H)$ be the group of automorphisms of the datum (H, κ, ξ) modulo inner automorphisms. It is a finite group. The following proposition is Proposition 10.9 of [9]. Set

$$c^S = \prod_{v \notin S} c_v,$$

where c_v is defined in §11.8.

Proposition 11.3. *One has*

$$J^H(f^H) = |\mathrm{Out}_G(H)| c^S \tau(H) \tau(G^*)^{-1} J^G(f).$$

Proof. Start with (10.3) in §10.5:

$$J^G(f) = \tau(G^*) \sum_{X^* \in \Sigma_{\mathrm{ell}}(\mathfrak{g}^*)} \sum_{\kappa \in \mathcal{K}(T_{X^*}, G^*, F)} I_{X^*}^{G, \kappa}(f).$$

Let $X^* \in \Sigma_{\mathrm{ell}}(\mathfrak{g}^*)$ and $\kappa \in \mathcal{K}(T_{X^*}, G^*, F)$. Let (H', κ', ξ') be an elliptic endoscopic datum for G^* and $Y \in \Sigma(\mathfrak{h}')$ such that $Y \rightarrow X^*$ and $\kappa_Y = \kappa$, as in Lemma 10.3. By §10.8,

$$I_{X^*}^{G, \kappa}(f) = \prod_{v \in V} I_Y^{G_v, H'_v}(f_v).$$

As usual, almost all factors in the infinite products under consideration are equal to 1. In what follows, suppose that $I_{X^*}^{G, \kappa}(f) \neq 0$.

This implies that $I_Y^{G_v, H'_v}(f_v) \neq 0$ for every $v \notin S$. But f_v is the characteristic function of a hyperspecial lattice. It follows that H'_v is unramified; this is a version of a lemma of Kottwitz, see [9, Lemma 7.4]. Furthermore, for $v \in S_2$, the nonvanishing of $I_Y^{G_v, H'_v}(f_v)$ implies that the centralizer of Y is split, by condition 1 of Proposition 11.2 and condition 2 of §11.5. Thus H' is unramified outside S_1 .

We may therefore assume that $(H', \kappa', \xi') \in \mathcal{H}$. If it lies in $\mathcal{H}'$, then, at $v = v_{(H', \kappa', \xi')}$, the nonvanishing of $I_Y^{G_v, H'_v}(f_v)$ implies as above that H'_v is split, a contradiction. Hence $F[H'] \subset F[H]$.

We now use the place u . By condition 3 of Proposition 11.2 and Lemma 10.3, the data (H_u, κ, ξ) and (H'_u, κ', ξ') are equivalent over F_u . Condition 3 of §11.1 shows that this isomorphism extends to an F -isomorphism.

Let $\Sigma_G(\mathfrak{h})(X^*, \kappa)$ be the subset of $\Sigma_G(\mathfrak{h})$ consisting of Y such that $Y \rightarrow X^*$ and $\kappa_Y = \kappa$. If nonempty, this subset has cardinality $|\mathrm{Out}_G(H)|$, by Lemma 10.3. It follows that

$$\begin{aligned} J^G(f) &= \frac{\tau(G^*)}{|\mathrm{Out}_G(H)|} \sum_{X^* \in \Sigma_{\mathrm{ell}}(\mathfrak{g}^*)} \sum_{\kappa \in \mathcal{K}(T_{X^*}, G^*, F)} \\ &\quad \cdot \sum_{Y \in \Sigma_G(\mathfrak{h})(X^*, \kappa)} \prod_{v \in V} I_Y^{G_v, H'_v}(f_v). \end{aligned}$$

We make the analogous calculation for H . Since the support of f_u^H consists of G -regular elements, we may write

$$J^H(f^H) = \tau(H) \sum_{Y \in \Sigma_{G, \mathrm{ell}}(\mathfrak{h})} \sum_{\kappa \in \mathcal{K}(T_Y, H, F)} I_Y^{H, \kappa}(f^H).$$

Let $Y \in \Sigma_{G, \mathrm{ell}}(\mathfrak{h})$ and $\kappa \in \mathcal{K}(T_Y, H, F)$. In §11.6, we defined $\kappa_v \in \mathcal{K}(T_Y, H, F_v)$ for every place v . One checks that

$$I_Y^{H, \kappa}(f^H) = \prod_{v \in V} I_Y^{H, \kappa_v}(f_v^H).$$

If this product is nonzero, then $I_Y^{H, \kappa_u}(f_u^H) \neq 0$, which forces $\kappa_u = 1$ by condition 4 of Proposition 11.2. Since T_Y is F -elliptic and even F_u -elliptic, by the construction of f_u and condition 3 of §11.5, one readily sees, for example using Tate–Nakayama duality, that

$$\mathcal{K}(T_Y, H, \mathbb{A}_F/F) \longrightarrow \mathcal{K}(T_Y, H, F_u)$$

is injective. Consequently $\kappa = 1$. We obtain

$$J^H(f^H) = \tau(H) \sum_{Y \in \Sigma_{G, \text{ell}}(\mathfrak{h})} \prod_{v \in V} S_Y^{H_v}(f_v^H).$$

Using the transfer properties of f_v and f_v^H , and in particular the fundamental lemma, we write

$$J^H(f^H) = \tau(H) c^S \sum_{Y \in \Sigma_{G, \text{ell}}(\mathfrak{h})} \prod_{v \in V} I_Y^{G_v, H_v}(f_v).$$

To conclude, it is enough to use the fact that $\Sigma_{G, \text{ell}}(\mathfrak{h})$ is the disjoint union of the sets $\Sigma_{G, \text{ell}}(\mathfrak{h})(X^*, \kappa)$ as $X^* \in \Sigma_{\text{ell}}(\mathfrak{g}^*)$ and $\kappa \in \mathcal{K}(T_{X^*}, G^*, F)$ vary. $\square$

11.13. Refining the conditions at the archimedean places. We may further assume, by [9, Lemma 10.7], that the archimedean functions are chosen so that

1. for every $v \in V_\infty$,

$$I_{\varphi_v^{-1}(X_0^*)}(\widehat{f}_v) \neq 0;$$

2. if $X \in \mathfrak{g}^*(F)$ and, at every place v , $\varphi_v^{-1}(X_v)$ is conjugate under $G(\overline{F}_v)$ to an element of $\text{supp}(\widehat{f}_v)$, then X is stably conjugate to X_0^* ;
3. if $Y \in \mathfrak{h}(F)$ and, at every place v , Y is conjugate under $H(\overline{F}_v)$ to an element of $\text{supp}(\widehat{f}_v^H)$, then $Y \rightarrow X_0^*$.

Of course, at the archimedean places we continue to require f_v^H to be a transfer of f_v , so that $\widehat{f}_v^H$ is a transfer of $\widehat{f}_v$, by Theorem 9.2.

11.14. Using the Poisson summation formula. The formula gives the following identity for every $x \in G(\mathbb{A})$:

$$\sum_{X \in \mathfrak{g}(F)} f(\text{Ad}(x^{-1})X) = \sum_{X \in \mathfrak{g}(F)} \widehat{f}(\text{Ad}(x^{-1})X).$$

The shape of the support of f_u shows that the left-hand sum is effectively over $X \in \mathfrak{g}_{\text{ell}}(F)$. By the preceding paragraph, the same is true of the right-hand sum. Integrating over $G(F) \backslash G(\mathbb{A})^1$, using Theorem 10.1, gives

$$J^G(f) = J^G(\widehat{f}). \tag{11.2}$$

The same reasoning gives

$$J^H(f^H) = J^H(\widehat{f^H}). \tag{11.3}$$

11.15. Calculation of $J^G(\widehat{f})$. We have the following lemma.

Lemma 11.4. *One has*

$$J^G(\widehat{f}) = \tau(G^*) I_{X_0^*}^{\kappa_0}(\widehat{f}).$$

Proof. Apply pre-stabilization to $J^G(\widehat{f})$. Since we also have condition 2 of §11.13, we obtain

$$J^G(\widehat{f}) = \tau(G^*) \sum_{\kappa \in \mathcal{K}(T_{X_0^*}, G^*, F)} I_{X_0^*}^{\kappa}(\widehat{f}).$$

Let $\kappa \in \mathcal{K}(T_{X_0^*}, G^*, F)$ satisfy $I_{X_0^*}^\kappa(\widehat{f}) \neq 0$. Choose (H', κ', ξ') and $Y \in \mathfrak{h}'_{G\text{-reg}}(F)$ such that $Y \rightarrow X_0^*$ and $\kappa'_Y = \kappa$, as in Lemma 10.3. Then

$$I_{X_0^*}^\kappa(\widehat{f}) = \prod_{v \in V} I_Y^{G_v, H'_v}(\widehat{f}_v).$$

Since $I_Y^{G_u, H'_u}(\widehat{f}_u) \neq 0$, condition 5 of Proposition 11.2 gives $\kappa_{0,u} = \kappa_u$. As $T_{X_0^*}$ is both F -elliptic and F_u -elliptic, the localization map for endoscopic characters is injective, as explained above. Hence $\kappa_0 = \kappa$, as required. $\square$

11.16. Calculation of $J^H(\widehat{f^H})$. We have the following lemma.

Lemma 11.5. *One has*

$$J^H(\widehat{f^H}) = \tau(H) \sum_{Y \in \mathcal{Y}_0} \prod_{v \in V} S_Y^{H_v}(\widehat{f_v^H}).$$

Proof. Apply pre-stabilization to $J^H(\widehat{f^H})$. This gives

$$J^H(\widehat{f^H}) = \tau(H) \sum_{Y \in \mathcal{Y}} \sum_{\kappa \in \mathcal{K}(T_Y, H, F)} I_Y^\kappa(\widehat{f^H}).$$

Let $Y \in \mathcal{Y}$ and $\kappa \in \mathcal{K}(T_Y, H, F)$ satisfy $I_Y^\kappa(\widehat{f^H}) \neq 0$. We have

$$I_Y^\kappa(\widehat{f^H}) = \prod_{v \in V} I_Y^{\kappa_v}(\widehat{f_v^H}).$$

By condition 6 of Proposition 11.2, $Y \in \mathcal{Y}_0$ and $\kappa_u = 1$. The same injectivity argument gives $\kappa = 1$. The factors in the product are therefore stable orbital integrals. $\square$

11.17. Conclusion. Lemmas 11.4 and 11.5, Proposition 11.3, and identities (11.2) and (11.3) give

$$\sum_{Y \in \mathcal{Y}_0} \prod_{v \in V} S_Y^{H_v}(\widehat{f_v^H}) = c^S |\text{Out}_G(H)| I_{X_0^*}^{\kappa_0}(\widehat{f}). \quad (11.4)$$

Set

$$c^w = \prod_{\substack{v \in V \setminus V_\infty \\ v \neq w}} c_v.$$

Using the fundamental lemma, the properties of f_v at finite places $v \in S$ distinct from w , especially condition 7 of Proposition 11.2, and the properties at archimedean places, we obtain, for every $Y \in \mathcal{Y}_0$,

$$\prod_{v \in V} S_Y^{H_v}(\widehat{f_v^H}) = c^w S_Y^{H_w}(\widehat{f_w^H}) \prod_{\substack{v \in V \\ v \neq w}} I_Y^{G_v, H_v}(\widehat{f_v}).$$

Let $v \in V$ with $v \neq w$. Choose $X_0 \in \mathfrak{g}(F_v)$ such that $Y \rightarrow X_0$. Equivalently, X_0 and X_0^* have the same image in $\mathfrak{c}$. In particular, this condition does not depend on the choice of $Y \in \mathcal{Y}_0$. For every $X \in \mathfrak{g}(F_v)$ such that $Y \rightarrow X$, formula (7.2) in §7.4 gives

$$\begin{aligned} \Delta_v(Y, X) &= \Delta_v(Y, X_0) \langle \kappa_{Y,v}, \text{inv}(X_0^*; X, X_0) \rangle \\ &= \Delta_v(Y, X_0) \langle \kappa_{0,v}, \text{inv}(X_0^*; X, X_0) \rangle \\ &= \frac{\Delta_v(Y, X_0)}{\Delta_v(Y_0, X_0)} \Delta_v(Y_0, X). \end{aligned}$$

Consequently,

$$I_Y^{G_v, H_v}(\widehat{f}_v) = \frac{\Delta_v(Y, X_0)}{\Delta_v(Y_0, X_0)} I_{Y_0}^{G_v, H_v}(\widehat{f}_v).$$

Using the product formula for transfer factors in §10.7, we obtain

$$\prod_{\substack{v \in V \\ v \neq w}} I_Y^{G_v, H_v}(\widehat{f}_v) = \frac{\Delta_w(Y_0, X_0)}{\Delta_w(Y, X_0)} \prod_{\substack{v \in V \\ v \neq w}} I_{Y_0}^{G_v, H_v}(\widehat{f}_v).$$

Lemma 11.6. *For every $Y \in \mathcal{Y}_0$, one has*

$$S_{Y_0}^{H_w}(\widehat{f_w^H}) = \frac{\Delta_w(Y_0, X_0)}{\Delta_w(Y, X_0)} S_Y^{H_w}(\widehat{f_w^H}).$$

Proof. We sketch the proof; for further details, see [9, Section 3]. Since $\kappa_Y = \kappa_{Y_0}$, there is an automorphism i of (H, κ, ξ) carrying Y_0 to Y , by Lemma 10.3. One shows, as in [9, Lemma 2.4], that there is a constant $\delta \neq 0$ such that, for every $Z \in \mathfrak{h}_{G\text{-reg}}(F_w)$ and every $X \in \mathfrak{g}(F_w)$ with $Z \rightarrow X$,

$$\Delta_w(i(Z), X) = \delta \Delta_w(Z, X). \quad (11.5)$$

In particular, since f_w^H is a transfer of f_w , for every $Z \in \mathfrak{h}_{G\text{-reg}}(F_w)$ we obtain

$$S_{i(Z)}^{H_w}(f_w^H) = \delta S_Z^{H_w}(f_w^H).$$

Next observe that i preserves the form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}(F_w)$. Thus it also preserves the functions $\widehat{i}^{H_w}$. Using Theorem 6.3, we conclude that

$$S_{i(Z)}^{H_w}(\widehat{f_w^H}) = \delta S_Z^{H_w}(\widehat{f_w^H})$$

for every $Z \in \mathfrak{h}_{G\text{-reg}}(F_w)$. To finish, evaluate δ by taking $(Z, X) = (Y_0, X_0)$ in (11.5). $\square$

By the preceding lemma,

$$\prod_{v \in V} S_Y^{H_v}(\widehat{f_v^H}) = c^w S_{Y_0}^{H_w}(\widehat{f_w^H}) \prod_{\substack{v \in V \\ v \neq w}} I_{Y_0}^{G_v, H_v}(\widehat{f}_v).$$

In particular, this product does not depend on Y . Identity (11.4) becomes

$$c^w S_{Y_0}^{H_w}(\widehat{f_w^H}) \prod_{\substack{v \in V \\ v \neq w}} I_{Y_0}^{G_v, H_v}(\widehat{f}_v) = c^S I_{X_0^*}^{\kappa_0}(\widehat{f}).$$

Since we also have

$$I_{X_0^*}^{\kappa_0}(\widehat{f}) = \prod_{v \in V} I_{Y_0}^{G_v, H_v}(\widehat{f}_v),$$

it follows that

$$\begin{aligned} & (c^w S_{Y_0}^{H_w}(\widehat{f_w^H}) - c^S I_{Y_0}^{G_w, H_w}(\widehat{f}_w)) \\ & \cdot \prod_{\substack{v \in V \\ v \neq w}} I_{Y_0}^{G_v, H_v}(\widehat{f}_v) = 0. \end{aligned}$$

The choices were made precisely to ensure that the second factor is nonzero. Hence

$$c^w S_{Y_0}^{H_w}(\widehat{f_w^H}) = c^S I_{Y_0}^{G_w, H_w}(\widehat{f}_w).$$

This equality also holds at Y_w , by the choice of Ω_w in §11.4, proving (11.1).

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